

A Converse Hawking-Unruh Effect and dS^2 /CFT Correspondence

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ABSTRACT. Given a local quantum field theory net \mathcal{A} on the de Sitter space-time dS^d , where geodesic observers are thermalized at Gibbons-Hawking temperature, we look for observers that feel to be in a ground state, i.e. particle evolutions with positive generator, providing a sort of converse to the Hawking-Unruh effect. Such positive energy evolutions always exist as noncommutative flows, but have only a partial geometric meaning, yet they map localized observables into localized observables.

We characterize the local conformal nets on dS^d . Only in this case our positive energy evolutions have a complete geometrical meaning. We show that each net has a unique maximal expected conformal subnet, where our evolutions are thus geometrical.

In the two-dimensional case, we construct a holographic one-to-one correspondence between local nets \mathcal{A} on dS^2 and local conformal non-isotonic families (pseudonets) \mathcal{B} on S^1 . The pseudonet \mathcal{B} gives rise to two local conformal nets \mathcal{B}_\pm on S^1 , that correspond to the \mathfrak{H}_\pm horizon components of \mathcal{A} , and to the chiral components of the maximal conformal subnet of \mathcal{A} . In particular, \mathcal{A} is holographically reconstructed by a single horizon component, namely the pseudonet is a net, iff the translations on \mathfrak{H}_\pm have positive energy and the translations on \mathfrak{H}_\mp are trivial. This is the case iff the one-parameter unitary group implementing rotations on dS^2 has positive/negative generator.

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1. INTRODUCTION

The thermalization effects discovered by Hawking [31], resp. by Unruh [54], have shown the concept of particle to be gravity, resp. observer, dependent; the two effects being related by Einstein equivalence principle.

Unruh effect deals in particular with a quantum field theory on Minkowski spacetime: an observer O with uniform acceleration a feels, in its proper Rindler spacetime W , the Hawking temperature $T_H = \frac{a}{2\pi}$. As noticed in [50], this can be also explained by the Bisognano-Wichmann theorem [2]: the one-parameter automorphism group describing the evolution of O in its proper observable algebra $\mathcal{A}(W)$ satisfies the KMS thermal equilibrium condition at inverse temperature $\beta_H = T_H^{-1}$, see [30].

The Gibbons-Hawking effect [22] occurs in the de Sitter spacetime with radius ρ . Here every inertial observer O feels the temperature $T_{GH} = \frac{1}{2\pi\rho}$. Again we may express this fact by saying that the one-parameter automorphism group describing the evolution of O in its proper observable algebra $\mathcal{A}(W)$ satisfies the KMS thermal equilibrium condition at inverse temperature $\beta_{GH} = T_{GH}^{-1}$, where W is here the static de Sitter spacetime ([20] for the two-dimensional case).

In the Minkowski spacetime, the KMS property for a uniformly accelerated observer O can be taken as a first principle, then the basic structure follows, in particular the Poincaré symmetries with positive energy are then derived [24, 11, 25, 29] and [16, 13] for a related approach. See also [29, 40] for weaker thermal conditions.

In the de Sitter space, the KMS property for the geodesic observer O can be taken as a first principle [6, 9, 44]; in particular the value of T_{GH} is then fixed automatically [6].

We mention at this point that actual observations in cosmology indicates that, on a large scale, our universe is isotropic, homogeneous and repulsively expanding. The de Sitter spacetime thus provides a good approximation model, at least asymptotically. Hence de Sitter spacetime and, more generally, Robertson-Walker spacetimes with positive scalar curvature are basic objects to be studied.

The first aim of this paper is to study a sort of converse to the above mentioned thermalization effects. Starting with the curved de Sitter space, where a geodesic observer is thermalized, we wish to find a different observer whose quantum evolution has positive generator, namely feels to be in a ground state. In other words, we want to keep the same state, but choose a time evolution w.r.t. which the state becomes a ground state. Now an observer in dS^d whose world line is an orbit of a boost experiences a temperature

$$T = \frac{1}{2\pi} \sqrt{\frac{1}{\rho^2} + a^2} \geq T_{GH},$$

[44], with a the modulus of the intrinsic uniform acceleration, contrary to our aim.

The dethermalization effect to take place is indeed a non-trivial matter. To understand this point notice that we are looking for a particle whose acceleration compensates the curvature of the underlying space so that, at least locally, the particle's picture of the spacetime is flat. Yet the particle's acceleration is a vector, but the curvature is a tensor so that, even in the constant scalar curvature case, there is no obvious way to fulfil the above requirement.

Indeed, it turns out that this cooling down effect is linked to the conformal invariance and, in two spacetime dimensions, to a holography in a sense similar to the one studied in the anti-de Sitter spacetime [43], as we shall explain.

As is known, Minkowski spacetime M^d is conformal to a double cone in the Einstein static universe E^d . On the other hand dS^d is conformal to a rectangular strip of E^d . Using this fact one can directly set up a bijective correspondence

between local conformal nets on M^d and on dS^d . Less obviously, this sets up a correspondence between positive energy-momentum local conformal nets on M^d and local conformal nets on dS^d with the KMS property for geodesic observers.

At this point it is immediate that, given any local, conformal, KMS geodesic net on dS^d , there exists a timelike conformal geodesic flow μ on dS^d that gives rise to a quantum evolution with positive generators: they are simply the ones that correspond to timelike translations on M^d . Let us remark that μ has only a local action on dS^d , namely in general $\mu_t x$ “goes outside” dS^d for large t .

We may ask whether the flow μ promotes to a ground state quantum evolution for a general local net on dS^d . In a sense we need to proceed similarly to Mechanics when one passes from a passive description (in terms of coordinates) to an active description (in terms of tensors). The answer is yes, but the evolution is only partially geometric. We shall show that there exists a one-parameter unitary group V with positive generator such that, in particular,

$$V(t)\mathcal{A}(\mathcal{O})V(-t) = \mathcal{A}(\mu_t\mathcal{O})$$

for certain regions \mathcal{O} and for all $t \in \mathbb{R}$ such that $\mu_t\mathcal{O}$ is still in dS^d , and

$$V(t)\mathcal{A}(\mathcal{O})V(-t) \subset \mathcal{A}(\tilde{\mathcal{O}}'_t),$$

for all double cones \mathcal{O} contained in the steady state universe subregion \mathcal{N} of dS (or contained in the complement of \mathcal{N}), for a suitable double cone $\tilde{\mathcal{O}}'_t$ depending on \mathcal{O} and t , cf. Remark 4.11. The unitary group V is constructed by the Borchers-Wiesbrock methods [4, 5, 55] and, in the conformal case, coincides with the previously considered one where the geometric meaning is complete.

Our analysis then proceeds to determine the maximal subnet of \mathcal{A} where the geometric meaning is complete. For any net \mathcal{A} we show that there exists a unique maximal expected conformal subnet, and this net has the required property.

Finally we consider more specifically the case of a two-dimensional de Sitter spacetime, with the aim of describing a local net on dS^2 via holographic reconstruction, namely in terms of a suitable conformal theory on S^1 . For different approaches to dS/CFT correspondence in the two-dimensional or in the higher dimensional case, see e.g. [51].

To this end we introduce the notion of pseudonet on S^1 . This is a family of local von Neumann algebras associated with intervals of S^1 where isotony is *not* assumed. Moreover, we assume Möbius covariance, commutativity between the algebra of an interval and that of its complement, the existence of an invariant cyclic (vacuum) vector, and the geometric meaning of the modular groups.

We shall show that a local conformal pseudonet \mathcal{B} encodes exactly the same information of a $SO_0(2,1)$ -covariant local net \mathcal{A} on dS^2 with the geodesic KMS property, namely we have a bijective correspondence, holography,

$$SO_0(2,1) - \text{covariant local nets on } dS^2 \leftrightarrow \text{local conformal pseudonets on } S^1.$$

The pseudonet naturally lives on one component \mathfrak{H}_+ or \mathfrak{H}_- of the cosmological horizon (choosing the other horizon component would amount to pass to the conjugate pseudonet), and the holographic reconstruction is based on a 1 : 1 geometric correspondence between wedges in dS^2 and their projections on \mathfrak{H}_\pm . Conformal invariance and chirality may then be described in terms of the pseudonet.

A net \mathcal{A} on dS^2 gives, by restriction, two nets \mathcal{A}_\pm on \mathfrak{H}_\pm , that turn out to be conformal, hence \mathcal{A}_\pm extend to conformal nets on S^1 . Then $\mathcal{A}_\chi := \mathcal{A}_+ \otimes \mathcal{A}_-$ is a two-dimensional chiral conformal net. It turns out that \mathcal{A}_χ is naturally identified with a conformal subnet of \mathcal{A} , indeed it is the chiral subnet of the maximal conformal subnet of \mathcal{A} .

From a different point of view, the pseudonet \mathcal{B} gives naturally rise to a pair of local conformal (i.e. Möbius covariant) nets \mathcal{B}_\pm on S^1 that correspond to \mathcal{A}_\pm .

Finally we address the question of when a net on dS^2 is holographically reconstructed by a conformal net, namely when the associated pseudonet is indeed isotonic. Let τ be the Killing flow (2.2) which restricts to the translations on \mathfrak{H}_+ , and U is the associated one-parameter unitary subgroup of the de Sitter group representation. Then U has positive generator if and only if \mathcal{B} is isotonic. This is perhaps a point where the relation between the dethermalization effect, conformal invariance and holography is more manifest. Indeed the two-dimensional case is the only case where the de Sitter group admits positive energy representations. This means that a massless particle on \mathfrak{H}_+ may evolve according with the flow τ (cf. subsection 2.1.4) hence feels a dethermalized vacuum if the representation is positive energy. However this is exactly the case where the net is conformal and “lives on \mathfrak{H}_+ ”, namely the restriction \mathcal{A}_- is trivial. Analogous result holds replacing \mathfrak{H}_+ with \mathfrak{H}_- .

2. GENERAL STRUCTURE

2.1. Geometrical preliminaries. We begin to recall some basic structure, mainly geometrical aspects, that will undergo our analysis.

2.1.1. Expanding universes and Gibbons-Hawking effect. As is known [22], a spacetime \mathcal{M} with repulsive (i.e. positive) cosmological constant has certain similarities with a black hole spacetime. \mathcal{M} is expanding so rapidly that, if γ is a freely falling observer in \mathcal{M} , there are regions of \mathcal{M} that are inaccessible to γ , even he waits indefinitely long; in other words the past of the world line of γ is a proper subregion \mathcal{N} of \mathcal{M} . The boundary \mathfrak{H} of \mathcal{N} is a cosmological event horizon for γ . As in the black hole case, one argues that γ detects a temperature related to the surface gravity of \mathfrak{H} . This is a quantum effect described by quantum fields on \mathcal{M} (see below); heuristically: spontaneous particle pairs creation happens on \mathfrak{H} , negative energy particles may tunnel into the inaccessible region, the others contribute to the thermal radiations.

2.1.2. de Sitter spacetime. The spherically symmetric, complete vacuum solution of Einstein equation with cosmological constant $\Lambda > 0$ is dS^d , the d -dimensional de Sitter spacetime. By Hopf theorem, if $d > 2$, dS^d is the unique complete simply connected spacetime with constant curvature $R = 2d\Lambda/(d-2)$ (if $d = 2$ this characterizes the universal covering of dS^2). dS^d may be defined as a pseudosphere, namely the submanifold of the ambient Minkowski spacetime \mathbb{R}^{d+1}

$$x_0^2 - x_1^2 - \dots - x_d^2 = -\rho^2$$

where the de Sitter radius is $\rho = \sqrt{\frac{(d-1)(d-2)}{2\Lambda}}$. dS^d is maximally symmetric, isotropic and homogeneous; the de Sitter group $SO(d, 1)$ acts transitively by isometries of dS^d . The geodesics of dS^d are obtained by intersecting dS^d with two-dimensional planes through the origin of \mathbb{R}^{d+1} , see e.g. [46, 45]. In particular the world line of a material freely falling observer is a boost flow line, say

$$(2.1) \quad \begin{cases} x_0(t) = \rho \sinh t \\ x_1(t) = \rho \cosh t \\ x_2(t) = 0 \\ \dots \\ x_d(t) = 0 \end{cases}$$

whose past is the steady state universe, the part of dS^d lying in the region $\mathcal{N} = \{x : x_1 > x_0\}$ and the cosmological horizon \mathfrak{H} is the intersection of dS^d with the plane $\{x : x_0 = x_1\}$.

The orbit of uniformly accelerated observers are obtained by intersecting dS^d with arbitrary planes of \mathbb{R}^{d+1} [47], of course only timelike and lightlike sections describe material and light particles, the others have constant imaginary acceleration.

2.1.3. Killing flows. We briefly recall a few facts about the proper spacetime and the corresponding evolution of an observer. Let \mathcal{M} be a Lorentzian manifold and $\gamma : \mathbb{R} \rightarrow \mathcal{M}$ a timelike or lightlike geodesic. The proper spacetime of the observer associated with γ is the causal completion W of γ . The relative acceleration of nearby particles is measured by the second derivative of the variation vector field V on γ ; by definition, if $x : \mathbb{R} \times (-\delta, \delta) \rightarrow \mathcal{M}$ is a smooth map with $\gamma(u) = x(u, 0)$ then $V(u) \equiv \partial_v x(u, v)|_{v=0}$. If x is geodesic, namely every map $u \rightarrow x(u, v)$ is a geodesic, then V is a Jacobi vector field, namely $V'' = R_{V\gamma'}\gamma'$ where R is the curvature tensor, showing that in general there is a non-zero tidal force $R_{V\gamma'}\gamma'$ (we use proper time parametrization in the timelike case).

On the other hand, if all maps $u \rightarrow x(u, v)$ are flow lines of a Killing flow τ , and $x(u, v) = \tau_u(x(0, v))$, then the tidal forces vanish. Indeed $V(u)$ is the image of $V(0)$ under the differential of τ_u , thus $V(u)$ parallel to $V(0)$ because γ is geodesic. Therefore the relative velocity, hence the relative acceleration, is 0.

In other words a Killing flow having the geodesic γ as a flow line describes an evolution which is static with respect to the freely falling particle associated with γ . We shall consider, in particular, the case where \mathcal{M} is dS^d and γ is a boost line; then W is, by definition, a *wedge* and the evolution associated with γ is described by the same one-parameter subgroup of the de Sitter group.

2.1.4. The two Killing flows of a lightlike particle. We consider now a null geodesic γ in dS^d . It lies in a section by a two-dimensional plane of \mathbb{R}^{d+1} through the origin that contains a lightlike vector of dS^d . By the transitivity of the $SO_0(d, 1)$ -action, we may assume the plane is $\{x : x_0 = x_1, x_2 = \dots = x_n = 0\}$.

Contrary to the timelike geodesic situation (2.1), which is the flow of a unique Killing flow (boosts), there are here two possible Killing flows with an orbit in this section. As γ is lightlike, there is no proper time associated with γ . We may parametrize γ , for example, as $\gamma_1(s) = x(s)$ with

$$(2.2) \quad \begin{cases} x_0(s) = s \\ x_1(s) = s \\ x_2(s) = 0 \\ \dots \\ x_d(s) = 0, \quad s \in \mathbb{R}, \end{cases}$$

or $\gamma_2(t) = x(t)$ with

$$(2.3) \quad \begin{cases} x_0(t) = e^t \\ x_1(t) = e^t \\ x_2(t) = 0 \\ \dots \\ x_d(t) = 0, \quad t \in \mathbb{R}, \end{cases}$$

namely $\gamma_2(t) = \gamma_1(e^t)$. The supports of the two curves are of course different, one is properly contained in the other: in the first case it is the entire line, while in the second case it is only a half-line.

Now $s \rightarrow \gamma_2(s)$ is a flow line of the boosts (2.1), the observable algebra is $\mathcal{A}(W)$ with W the wedge as above, and we are in the Hawking-Unruh situation. The boundary of W is a “black hole” horizon for the boosts: the observer associated with γ_2 cannot send a signal out of W and get it back.

Also $t \rightarrow \gamma_1(t)$ is the flow line of a Killing flow τ . If $d = 2$ we may use the usual identification of a point $(x_0, x_1, x_2) \in \mathbb{R}^3$ with the matrix $\tilde{x} = \begin{pmatrix} x_0+x_2 & x_1 \\ x_1 & x_0-x_2 \end{pmatrix}$, so that the determinant of \tilde{x} is the square of the Lorentz length of x . Now $PSL(2, \mathbb{R})$ acts on \mathbb{R}^3 by the adjoint map $A \in PSL(2, \mathbb{R}) \mapsto \text{Ad } A \in SO(2, 1)$ where

$$\text{Ad } A : \tilde{x} \rightarrow A\tilde{x}A^T.$$

The map Ad is an isomorphism of $PSL(2, \mathbb{R})$ with $SO_0(2, 1)$, the connected component of the identity of $SO(2, 1)$, (we often identify $PSL(2, \mathbb{R})$ and $SO_0(2, 1)$) and the flow τ is given by

$$(2.4) \quad \tau_t = \text{Ad} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : \tilde{x} \rightarrow \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \tilde{x} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

(If $d > 2$, τ is given by the same formula, but acts trivially on the x_k coordinate, $k > 2$.)

Proposition 2.1. *The above flow τ is the unique Killing flow having the curve (2.2) as a flow line. τ is lightlike on \mathfrak{H} and otherwise spacelike.*

Proof. The statement is proved by elementary computations. \square

In this case the observable algebra of γ_2 is $\mathcal{A}(\mathcal{N}) = \mathcal{A}(dS^d)$ ($= B(\mathcal{H})$ in the irreducible case), and τ acts on the cosmological event horizon $\mathfrak{H} = \{x : x_1 = x_0\}$, the boundary of $\mathcal{N} = \{x : x_1 > x_0\}$.

2.2. Quantum fields and local algebras. So far we have mainly discussed geometrical aspects of dS^d . We now consider a quantum field on dS^d , but we assume that back reactions are negligible, namely the geometry of dS^d is not affected by the field.

Let us denote by \mathcal{K} the set of *double cones* of dS^d , namely \mathcal{K} is the set of non-empty open regions of dS^d with compact closure that are the intersection of the future of x and the past of y , where $x, y \in dS^d$ and y belongs to the future of x . A wedge is the limit case where x and y go to infinity. We shall denote by \mathcal{W} the set of wedges and by $\tilde{\mathcal{K}}$ the set of double cones, possibly with one or two vertex at infinity, thus $\tilde{\mathcal{K}} \supset \mathcal{K} \cup \mathcal{W}$. Elements of $\tilde{\mathcal{K}}$ are obtained by intersecting a family of wedges.

The field is described by a (local) net \mathcal{A} with the following properties.

a) Isotony and locality. \mathcal{A} is an *inclusion preserving* map

$$(2.5) \quad \mathcal{O} \in \mathcal{K} \rightarrow \mathcal{A}(\mathcal{O})$$

from double cones $\mathcal{O} \subset dS^d$ to von Neumann algebras $\mathcal{A}(\mathcal{O})$ on a fixed Hilbert space \mathcal{H} . $\mathcal{A}(\mathcal{O})$ is to be interpreted as the algebra generated by all observables which can be measured in \mathcal{O} .

For a more general region $D \subset dS^d$ the algebra $\mathcal{A}(D)$ is defined as the von Neumann algebra generated by the local algebras $\mathcal{A}(\mathcal{O})$ with $\mathcal{O} \subset D$, $\mathcal{O} \in \mathcal{K}$.

The local algebras are supposed to satisfy the condition of *locality*, i.e.

$$(2.6) \quad \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)' \text{ if } \mathcal{O}_1 \subset \mathcal{O}_2',$$

where \mathcal{O}' denotes the spacelike complement of \mathcal{O} in dS and $\mathcal{A}(\mathcal{O})'$ the commutant of $\mathcal{A}(\mathcal{O})$ in $\mathcal{B}(\mathcal{H})$.

b) Covariance. There is a continuous unitary representation U of the de Sitter group $SO_0(d, 1)$ on \mathcal{H} such that for each region $\mathcal{O} \subset dS^d$

$$(2.7) \quad U(g)\mathcal{A}(\mathcal{O})U(g)^{-1} = \mathcal{A}(g\mathcal{O}), \quad g \in SO_0(d, 1).$$

c) Vacuum with geodesic KMS property. There is a unit vector $\Omega \in \mathcal{H}$, the vacuum vector, which is U -invariant and cyclic for the global algebra $\mathcal{A}(dS^d)$. The corresponding vector state ω given by

$$(2.8) \quad \omega(A) = (\Omega, A\Omega),$$

has the following geodesic KMS-property (see [6]): For every wedge W the restriction $\omega|_{\mathcal{A}(W)}$ satisfies the KMS-condition at some inverse temperature $\beta > 0$ with respect to the time evolution (boosts) $\Lambda_W(t)$, $t \in \mathbb{R}$, associated with W . In other words, for any pair of operators $A, B \in \mathcal{A}(W)$ there exists an analytic function F in the strip $D \equiv \{z \in \mathbb{C} : 0 < \text{Im} z < \beta\}$, bounded and continuous on the closure \bar{D} of D , such that

$$(2.9) \quad F(t) = \omega(A\alpha_t(B)), \quad F(t + i\beta) = \omega(\alpha_t(B)A), \quad t \in \mathbb{R},$$

where $\alpha_t = \text{Ad } U(\Lambda_W(t))$.

d) Weak additivity. For each open region $\mathcal{O} \subset dS$ we have

$$(2.10) \quad \bigvee_{g \in SO_0(d, 1)} \mathcal{A}(g\mathcal{O}) = \mathcal{A}(dS^d),$$

where the lattice symbol \vee denotes the generated von Neumann algebra.

Proposition 2.2. [6, Borchers-Buchholz] *The following hold:*

- Reeh-Schlieder property: Ω is cyclic for $\mathcal{A}(\mathcal{O})$ for each fixed open non-empty region \mathcal{O} (hence it is separating for $\mathcal{A}(\mathcal{O})$ if the interior of \mathcal{O}' is non-empty).
- Wedge duality: For each wedge W we have $\mathcal{A}(W)' = \mathcal{A}(W')$.
- Gibbons-Hawking temperature: The inverse temperature is $\beta = 2\pi\rho$.
- PCT symmetry: The representation U of $SO_0(d, 1)$ extends to a (anti-) unitary representation of $SO(d, 1)$ acting covariantly on \mathcal{A} .

The Reeh-Schlieder property is obtained by using the KMS property in place of the analyticity due to the positivity of the energy in the usual argument in the Minkowski space.

Wedge duality then follows as usual by the geometric action of the modular group due to Takesaki theorem; that is to say, if $D = W$ is a wedge and $\mathcal{L} \subset \mathcal{A}(W)$ is a von Neumann algebra cyclic on Ω and globally stable under $\text{Ad } \Lambda_W$, then $\mathcal{L} = \mathcal{A}(W)$; this is a known fact in Minkowski spacetime, see e.g. [11]. Note that this argument also shows that the definition of the von Neumann algebra $\mathcal{A}(W)$ is univocal if W is a wedge.

Concerning the construction of the PCT anti-unitary, a corresponding result in the Minkowski space is contained in [25].

Lemma 2.3. *Let \mathcal{A} satisfy a), b) and c). Then \mathcal{A} is weakly additive iff the Reeh-Schlieder property holds.*

Proof. Because of Prop. 2.2 it is sufficient to show that if $\mathcal{O} \in \mathcal{K}$ and $\overline{\mathcal{A}(\mathcal{O})\Omega} = \mathcal{H}$ then the von Neumann algebra \mathcal{L} generated by the union of $\mathcal{A}(g\mathcal{O})$, $g \in SO_0(d, 1)$, is equal to $\mathcal{A}(dS^d)$.

Now $\mathcal{L} \supset \vee_t \mathcal{A}(\Lambda_W(t)\mathcal{O})$ and the latter is equal $\mathcal{A}(W)$ by Takesaki theorem. Thus $\mathcal{L} \supset \mathcal{A}(W_1)$ for every wedge W_1 because $SO_0(d, 1)$ acts transitively on \mathcal{W} and we conclude $\mathcal{L} = \mathcal{A}(dS^d)$ because every double cone is contained in a wedge. \square

It follows as in [25, Prop. 3.1] (see also [6]) that the center \mathcal{Z} of $\mathcal{A}(W)$ coincides with the center of $\mathcal{A}(dS^d)$ and \mathcal{A} has a canonical disintegration, along \mathcal{Z} , into (almost everywhere) irreducible nets. Moreover \mathcal{A} is *irreducible*, i.e. $\mathcal{A}(dS^d) = B(\mathcal{H})$, if and only if Ω is the *unique U -invariant vector* (see also [26]).

We shall say that \mathcal{A} satisfies *Haag duality* if

$$\mathcal{A}(\mathcal{O})' = \mathcal{A}(\mathcal{O}')$$

for all double cones $\mathcal{O} \in \mathcal{K}$.

Now it is elementary to check that every double cone is an intersection of wedges, indeed

$$\mathcal{O} = \bigcap \mathcal{W}_{\mathcal{O}}, \quad \mathcal{O} \in \tilde{\mathcal{K}},$$

where $\mathcal{W}_{\mathcal{O}}$ denotes the set of wedges containing \mathcal{O} .

We then define the *dual net* $\hat{\mathcal{A}}$ as

$$\hat{\mathcal{A}}(\mathcal{O}) \equiv \bigcap_{W \in \mathcal{W}_{\mathcal{O}}} \mathcal{A}(W),$$

Note that $\hat{\mathcal{A}}(W) = \mathcal{A}(W)$ if W is a wedge, hence $\hat{\mathcal{A}}(D) = \mathcal{A}(D)$ if every double cone $\mathcal{O} \subset D$ is contained in a wedge $W \subset D$ (this is the case if D is union of wedges).

By wedge duality the net $\hat{\mathcal{A}}$ is local (two spacelike separated double cones are contained in two spacelike separated wedges) and satisfies all properties **a)**-**d)**.

The following proposition is the version of a known fact in Minkowski space, cf. [48].

Proposition 2.4. *$\hat{\mathcal{A}}$ is Haag dual:*

$$\hat{\mathcal{A}}(\mathcal{O})' = \hat{\mathcal{A}}(\mathcal{O}') (= \mathcal{A}(\mathcal{O}')), \quad \mathcal{O} \in \mathcal{K},$$

and $\mathcal{A} = \hat{\mathcal{A}}$ iff \mathcal{A} satisfies Haag duality.

Proof. Let $\{W_i\}$ be the set of wedges in $\mathcal{W}_{\mathcal{O}}$. Then $\hat{\mathcal{A}}(\mathcal{O}) = \bigcap_i \hat{\mathcal{A}}(W_i)$, hence $\hat{\mathcal{A}}(\mathcal{O})' = (\bigcap_i \mathcal{A}(W_i))' = \bigvee_i \mathcal{A}(W_i') \subset \hat{\mathcal{A}}(\mathcal{O}') \subset \hat{\mathcal{A}}(\mathcal{O})'$.

To check the last part, it is sufficient to assume that \mathcal{A} satisfies Haag duality and show that $\mathcal{A} = \hat{\mathcal{A}}$. Indeed in this case $\mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{O}')' = (\bigvee_i \mathcal{A}(W_i'))' = \bigcap_i \mathcal{A}(W_i) = \hat{\mathcal{A}}(\mathcal{O})$. \square

If τ is a flow in dS^d , in general we may expect a corresponding quantum evolution only if τ is static, namely if τ is a Killing flow (see Subsect. 2.1.4). In this case there is a one-parameter unitary group U on \mathfrak{H} implementing τ covariantly:

$$U(t)\mathcal{A}(\mathcal{O})U(t)^* = \mathcal{A}(\tau_t\mathcal{O}).$$

Indeed U is a one-parameter subgroup of the unitary representation of $SO_0(d, 1)$, the connected component of the identity in $SO(d, 1)$.

In particular, if $\gamma : u \in \mathbb{R} \mapsto \gamma(u) \in dS^d$ is a timelike or lightlike geodesic, the evolution for the observer associated with γ is given by a Killing flow having γ in one orbit. Now the observable algebra associated with γ is $\mathcal{A}(W)$, where W is the causal envelope of γ , which is globally invariant with respect to τ . If γ describes a material particle, namely τ is a boost, then W is the wedge region globally invariant with respect to such boosts. By the geodesic KMS property, $\text{Ad}U$ is a one-parameter automorphism group of $\mathcal{A}(W)$ that satisfies the KMS thermal equilibrium condition at temperature $\frac{1}{2\pi\rho}$ [20, 9] and this corresponds, as is known, to the Hawking-Unruh effect [31, 54].

2.2.1. *Subnets.* Given a net \mathcal{A} on dS^d on a Hilbert space \mathcal{H} , namely \mathcal{A} satisfies properties **a**), **b**), **c**), **d**), we shall say the \mathcal{B} is a *subnet* of \mathcal{A} if

$$\mathcal{B} : \mathcal{O} \in \mathcal{K} \rightarrow \mathcal{B}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O})$$

is a isotonic map from double cones to von Neumann algebras such that

$$U(g)\mathcal{B}(\mathcal{O})U(g)^{-1} = \mathcal{B}(g\mathcal{O}), \quad g \in SO_0(1, d),$$

where U is the representation of the de Sitter group associated with \mathcal{A} . \mathcal{B} is extended to any region as above.

Clearly \mathcal{B} satisfies the properties **a**), **b**) and **c**), but for the cyclicity of Ω . By the Reeh-Schlieder theorem argument

$$(2.11) \quad \overline{(\bigvee_{g \in SO_0(d,1)} \mathcal{B}(g\mathcal{O}))\Omega} = \overline{\mathcal{B}(\mathcal{O})\Omega} \quad \mathcal{O} \in \tilde{\mathcal{K}},$$

where the bar denotes the closure, thus we have

$$(2.12) \quad \overline{\mathcal{B}(W)\Omega} = \mathcal{H}_{\mathcal{B}} \equiv \overline{\mathcal{B}(dS^d)\Omega}$$

for every $W \in \mathcal{W}$, because the de Sitter group acts transitively on \mathcal{W} and every double cone is contained in a wedge. Thus $\overline{\mathcal{B}(D)\Omega}$ is independent of the region $D \subset dS^2$ if D contains a wedge.

Clearly \mathcal{B} acts on $\mathcal{H}_{\mathcal{B}}$ and we denote by \mathcal{B}_0 its restriction to $\mathcal{H}_{\mathcal{B}}$. Note that \mathcal{B}_0 is net satisfying all properties **a**), **b**), **c**), but not necessarily **d**).

We shall say that a subnet \mathcal{B} is *expected* (in \mathcal{A}) if for every $\mathcal{O} \in \mathcal{K}$ there exists a vacuum preserving conditional expectation of $\mathcal{A}(\mathcal{O})$ onto $\mathcal{B}(\mathcal{O})$

$$\varepsilon_{\mathcal{O}} : \mathcal{A}(\mathcal{O}) \rightarrow \mathcal{B}(\mathcal{O})$$

such that

$$\varepsilon_{\mathcal{O}} \upharpoonright_{\mathcal{A}(\mathcal{O}_0)} = \varepsilon_{\mathcal{O}_0} \quad \mathcal{O}_0 \subset \mathcal{O}, \mathcal{O}_0, \mathcal{O} \in \tilde{\mathcal{K}}.$$

It is easily seen that that $\varepsilon_{\mathcal{O}}$ is given by

$$\varepsilon_{\mathcal{O}}(X)\Omega = E_{\mathcal{O}}X\Omega, \quad X \in \mathcal{A}(\mathcal{O}),$$

where $E_{\mathcal{O}}$ is the orthogonal projection onto $\overline{\mathcal{B}(\mathcal{O})\Omega}$.

Lemma 2.5. *If \mathcal{B} is expected, then \mathcal{B}_0 is weakly additive.*

Proof. Let \mathcal{O} be a double cone and $W \supset \mathcal{O}$ a wedge. If $X \in \mathcal{A}(\mathcal{O})$ we have

$$\varepsilon_{\mathcal{O}}(X)\Omega = E_{\mathcal{O}}X\Omega = \varepsilon_W(X)\Omega = E_WX\Omega = EX\Omega$$

where E is the projection on $\mathcal{H}_{\mathcal{B}}$, hence

$$\overline{\mathcal{B}(\mathcal{O})\Omega} = \overline{\varepsilon_{\mathcal{O}}(\mathcal{A}(\mathcal{O}))\Omega} = \overline{E\mathcal{A}(\mathcal{O})\Omega} = E\mathcal{H} = \mathcal{H}_{\mathcal{B}},$$

namely $\mathcal{B}_0(\mathcal{O})$ is cyclic on Ω . The statement then follows by Lemma 2.3. \square

The following Lemma is elementary, but emphasizes a property that holds on dS^d but not on M^d and makes a qualitative difference in the subnet analysis in the two cases.

Lemma 2.6. *Let \mathcal{O} be a double cone in dS^d . The the union of all wedges in $\mathcal{W}_{\mathcal{O}} \equiv \{W \in \mathcal{W} : W \supset \mathcal{O}\}$ has non-empty causal complement (it is the double cone antipodal to \mathcal{O}).*

Proof. If $x \in dS^d$, let \bar{x} denote its antipodal point. If W is a wedge, then W' is the antipodal of W , hence W contains x iff W' contains \bar{x} . If $\{W_i\}$ is a family of wedges, then

$$x \in \bigcap_i W_i \Leftrightarrow \bar{x} \in \bigcap_i W'_i = (\bigcup_i W_i)'.$$

Thus if $\bigcap_i W_i = \mathcal{O}$ the spacelike complement of $\bigcup_i W_i$ is the antipodal of \mathcal{O} . \square

Proposition 2.7. *Let \mathcal{A} be a local net on dS^d on a Hilbert space \mathcal{H} , and \mathcal{B} a subnet. Setting $\hat{\mathcal{B}}(\mathcal{O}) = \bigcap_{W \in \mathcal{W}_{\mathcal{O}}} \mathcal{B}(W)$, the following hold:*

- (i) $\hat{\mathcal{B}}$ restricts to $\widehat{\mathcal{B}}_0$ on $\mathcal{H}_{\mathcal{B}}$ (the dual net of \mathcal{B}_0).
- (ii) \mathcal{B}_0 is Haag dual iff $\mathcal{B} = \hat{\mathcal{B}}$.
- (iii) $\hat{\mathcal{B}}$ is an expected subnet of $\hat{\mathcal{A}}$.
- (iv) \mathcal{B} is expected in $\hat{\mathcal{A}}$ iff \mathcal{B}_0 is Haag dual.

Proof. (i): By Lemma 2.7 $D' \neq \emptyset$ where $D \equiv \cup\{W : W \in \mathcal{W}_{\mathcal{O}}\}$. Hence Ω is separating for $\mathcal{A}(D)$, so the map $X \in \mathcal{B}(D) \mapsto X \upharpoonright_{\mathcal{H}_{\mathcal{B}}}$ is an isomorphism between $\mathcal{B}(D)$ on \mathcal{H} and $\mathcal{B}(D)$ on $\mathcal{H}_{\mathcal{B}}$. It follows that the operation $\bigcap_{W \in \mathcal{W}_{\mathcal{O}}} \mathcal{B}(W)$ of taking intersection commutes with the restriction map.

(ii): By Prop. 2.4 \mathcal{B} is Haag dual iff $\widehat{\mathcal{B}}_0 = \mathcal{B}_0$, thus iff $\mathcal{B} = \hat{\mathcal{B}}$ by the previous point.

(iii): If W is a wedge then by the geodesic KMS property and Takesaki theorem there exists a vacuum preserving conditional expectation $\varepsilon_W : \mathcal{A}(W) \rightarrow \mathcal{B}(W)$ such that

$$\varepsilon_W(X)E = EXE, \quad X \in \mathcal{A}(W),$$

where E is the orthogonal projection onto $\overline{\mathcal{B}(W)\Omega} = \mathcal{H}_{\mathcal{B}}$, (cf. [18, 41]).

Let $X \in \hat{\mathcal{A}}(\mathcal{O})$ and $W \in \mathcal{W}_{\mathcal{O}}$. Since $X \in \mathcal{A}(W)$, we have $\varepsilon_W(X) \in \mathcal{B}(W)$, so there exists $Y_W \in \mathcal{B}(W)$ such that $Y_W E = EXE$. If W_1 is another wedge in $\mathcal{W}_{\mathcal{O}}$ then $Y_{W_1}\Omega = EX\Omega = Y_W\Omega$, thus $Y_{W_1} = Y_W$ because Ω is separating for $\mathcal{B}(W_1) \vee \mathcal{B}(W)$ by Lemma 2.6 and Reeh-Schlieder theorem. Thus the operator $Y \equiv Y_W$ is independent of $W \in \mathcal{W}_{\mathcal{O}}$ and belongs to $\mathcal{B}(W)$ for all wedges in $\mathcal{W}_{\mathcal{O}}$, namely $Y \in \hat{\mathcal{B}}(\mathcal{O})$. The map $X \mapsto Y$ is clearly a vacuum preserving conditional expectation from $\hat{\mathcal{A}}(\mathcal{O})$ onto $\hat{\mathcal{B}}(\mathcal{O})$.

(iv): If \mathcal{B}_0 , then $\mathcal{B} = \hat{\mathcal{B}}$ by (i) and \mathcal{B} is expected in $\hat{\mathcal{A}}$ by (iii).

Conversely, assume that \mathcal{B} is expected in $\hat{\mathcal{A}}$. We have

$$\hat{\mathcal{B}}(\mathcal{O}) = \bigcap_{W \in \mathcal{W}_{\mathcal{O}}} \mathcal{B}(W) \subset \bigcap_{W \in \mathcal{W}_{\mathcal{O}}} \mathcal{A}(W) = \hat{\mathcal{A}}(\mathcal{O}),$$

so, if $X \in \hat{\mathcal{B}}(\mathcal{O})$, then $X \in \hat{\mathcal{A}}(\mathcal{O})$ and $EXE = XE$, namely $\varepsilon_{\mathcal{O}}(X) = X$, so $X \in \mathcal{B}(\mathcal{O})$. Thus $\hat{\mathcal{B}} = \mathcal{B}$. \square

3. CONFORMAL FIELDS

3.1. Basics on the conformal structure. It is a known fact that several interesting spacetimes can be conformally embedded in the Einstein static universe, see [32, 1]. We shall recall here some embeddings and we begin with a discussion about conformal transformations.

3.1.1. The conformal group and the conformal completion. Two metrics on a manifold are said to belong to the same conformal class if one is a multiple of the other by a strictly positive function. Given two semi-Riemannian manifolds $\mathcal{M}_1, \mathcal{M}_2$, a *local conformal map* is a triple $(\mathcal{D}_1, \mathcal{D}_2, T)$ where $\mathcal{D}_1 \subset \mathcal{M}_1, \mathcal{D}_2 \subset \mathcal{M}_2$ are open, non-empty sets and $T : \mathcal{D}_1 \mapsto \mathcal{D}_2$ is a diffeomorphism which pulls back the metric on \mathcal{M}_2 to a metric in the same conformal class as the original metric on \mathcal{M}_1 .

With \mathcal{M} a d -dimensional semi-Riemannian manifold, a *conformal vector field* is a vector field Z on \mathcal{M} that satisfies the conformal Killing-Cartan equation: there exists a function f such that

$$(3.1) \quad \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle = f \langle X, Y \rangle,$$

for all vector fields X, Y, Z .

Conformal vector fields form a Lie algebra (the exponentiate to local one-parameter groups to local conformal maps, see below). We shall now assume $d \geq 3$ (our discussion will motivate definitions also in the $d = 2$ case). The dimension of the Lie algebra of the conformal Killing vector fields is then finite and indeed lower or equal than $(d+1)(d+2)/2$, the equality holding if and only if the manifold is conformally flat, namely the metric tensor is equal to the flat one up to a nonvanishing function [53]. Such Lie algebra is called the *conformal Lie algebra* of \mathcal{M} , and is denoted by $\text{conf}(\mathcal{M})$. Let us observe that such Lie algebra does not really depend on the metric on \mathcal{M} , but only on the conformal class, namely two metrics on \mathcal{M} in the same conformal class give rise to the same Lie algebra $\text{conf}(\mathcal{M})$.

Let us recall now that a Lie group G *acts locally* on a manifold \mathcal{M} if there exists an open set $W \subset G \times \mathcal{M}$ and a C^∞ map

$$(3.2) \quad T : W \rightarrow \mathcal{M}$$

$$(3.3) \quad (g, x) \mapsto T_g x$$

with the following properties:

- (i) $\forall x \in \mathcal{M}$, $V_x \equiv \{g \in G : (g, x) \in W\}$ is an open connected neighborhood of the identity $e \in G$;
- (ii) $T_e x = x$, $\forall x \in \mathcal{M}$;
- (iii) If $(g, x) \in W$, then $V_{T_g x} = V_x g^{-1}$ and moreover for any $h \in G$ such that $hg \in V_x$

$$T_h T_g x = T_{hg} x.$$

In general, a vector field satisfying equation (3.1) gives rise to a one-parameter family of (non-globally defined) transformations that are local conformal mappings, namely to a local action of \mathbb{R} on \mathcal{M} by means of local conformal maps, therefore $\text{conf}(\mathcal{M})$ exponentiates to a (connected, simply connected) group acting on \mathcal{M} by local conformal mappings. We shall call this Lie group the *local conformal group* of \mathcal{M} , and denote it by $\text{CONF}_{loc}(\mathcal{M})$.

A manifold \mathcal{M} is *conformally complete* if the elements of $\text{CONF}_{loc}(\mathcal{M})$ are everywhere defined maps, i.e. $\text{CONF}_{loc}(\mathcal{M})$ is contained in $\text{CONF}(\mathcal{M})$, the group of global conformal transformations of \mathcal{M} . Obviously, in this case $\text{CONF}_{loc}(\mathcal{M})$ is contained in the $\text{CONF}_0(\mathcal{M})$, the connected component of the identity in $\text{CONF}(\mathcal{M})$.

Lemma 3.1. *The stabilizer H of a point x under the action of the group $\text{CONF}_{loc}(\mathcal{M})$ is a closed subgroup.*

Proof. Let's prove the group property. Indeed if $g \in V_x$ stabilizes x then V_x is g -invariant, by (iii) above. Then, if $g, h \in V_x$ stabilize x , $h \in V_x = V_x g^{-1}$, namely $hg \in V_x$, and clearly $T_{hg} x = x$. Now assume $g_n \rightarrow g$, $g_n \in V_x$ and $g_n x = x$. Then there exists n_0 such that, for $n > n_0$, $g_n^{-1} g \in V_x$, therefore $g = g_n \cdot g_n^{-1} g \in V_x$, and $gx = x$ follows by continuity. \square

Let us assume that $\text{CONF}_{loc}(\mathcal{M})$ acts transitively on \mathcal{M} . We may therefore identify \mathcal{M} with an open subspace of the homogeneous space $\widetilde{\mathcal{M}} = \text{CONF}_{loc}(\mathcal{M})/H$. Clearly, the Lie algebra $\text{conf}(\widetilde{\mathcal{M}})$ coincides with the Lie algebra $\text{conf}(\mathcal{M})$, and $\text{CONF}_{loc}(\mathcal{M})$ acts globally on $\widetilde{\mathcal{M}}$. Therefore the local conformal group of \mathcal{M} (= the local conformal group of $\widetilde{\mathcal{M}}$) acts globally on $\widetilde{\mathcal{M}}$, namely $\widetilde{\mathcal{M}}$ is conformally complete.

Let us note that in general the action of $\text{CONF}_{loc}(\mathcal{M})$ may be non-effective on $\widetilde{\mathcal{M}}$, namely there may be non-identity elements of $\text{CONF}_{loc}(\mathcal{M})$ acting trivially. Therefore in general $\text{CONF}_0(\widetilde{\mathcal{M}})$ is a quotient of $\text{CONF}_{loc}(\mathcal{M})$.

Now we come back to the case of a non conformally complete manifold \mathcal{M} on which $\text{CONF}_{loc}(\mathcal{M})$ acts transitively, and suppose that there exists a discrete central subgroup Γ of $\text{CONF}(\widetilde{\mathcal{M}})$ such that \mathcal{M} is a fundamental domain for Γ , namely $\cup_{\gamma} \gamma \mathcal{M}$ is dense in $\widetilde{\mathcal{M}}$ and the $\gamma \mathcal{M}$'s are disjoint. Then $\widetilde{\mathcal{M}}/\Gamma$ is conformally complete and \mathcal{M} embeds densely in it. In this case, $\widetilde{\mathcal{M}}/\Gamma$ is denoted by $\overline{\mathcal{M}}$ and is called the conformal completion of \mathcal{M} . (In the cases we shall consider, and possibly in all cases, the choice of Γ is unique, thus the definition of $\overline{\mathcal{M}}$ does not depend on Γ).

We now summarize the construction of the conformal completion in the following diagram:

$$\begin{array}{ccccc}
 \mathcal{M} & \xrightarrow{\text{conf. Killing v. fields}} & \text{conf}(\mathcal{M}) & \xrightarrow{\text{exponential}} & \text{CONF}_{loc}(\mathcal{M}) \\
 \downarrow \text{completion} & & & & \downarrow \text{(transitive case) } H \text{ stabilizer} \\
 \overline{\mathcal{M}} & \xlongequal{\quad} & \widetilde{\mathcal{M}}/\Gamma & \xleftarrow{\Gamma \text{ discrete central}} & \widetilde{\mathcal{M}} = \text{CONF}_{loc}(\mathcal{M})/H
 \end{array}$$

Clearly $\text{CONF}(\overline{\mathcal{M}})$ acts on \mathcal{M} by restriction. Such action is indeed quasi global [10], namely the open set

$$\{x \in \mathcal{M} : (g, x) \in W\}$$

is the complement of a meager set S_g , and the following equation holds:

$$(3.4) \quad \lim_{x \rightarrow x_0} T_g x = \infty, \quad g \in G, \quad x_0 \in S_g,$$

where x approaches x_0 out of S_g and a point goes to infinity when it is eventually out of any compact subset of \mathcal{M} . It has been proved in [10] that any quasi-global action of a Lie group G on a manifold \mathcal{M} gives rise to a unique G -completion, namely to a unique manifold $\overline{\mathcal{M}}$ in which \mathcal{M} embeds densely and on which the action of G is global.

Now $\text{CONF}(\overline{\mathcal{M}})$ acts quasi-globally on \mathcal{M} , we shall follow the standard usage in physics and call it the *conformal group* of \mathcal{M} , denoting it by $\text{Conf}(\mathcal{M})$. Of course when \mathcal{M} is conformally complete $\text{CONF}(\mathcal{M}) = \text{Conf}(\mathcal{M})$.

In the $d = 2$ case, the Lie algebra of conformal Killing vector fields is infinite-dimensional. The above discussion goes through by considering a finite-dimensional sub-Lie group. In the Minkowski spacetime (an in conformally related spacetimes, see Sect. 3.2) this is the Lie algebra of the group generated by the Poincaré group and the ray inversion map.

Analogous considerations can be made for isometries, namely by replacing conformal Killing vector fields by Killing vector fields which are obtained setting $f = 0$ in equation (3.1). These gives rise to the local one-parameter groups with values in $\text{Iso}_{loc}(\mathcal{M})$, the local isometry group. If a Lorentzian manifold is geodesically complete then $\text{Iso}_{loc}(\mathcal{M})$ acts globally on it (cf. e.g. [36]).

3.1.2. The embedding of M^d . Einstein static universe $E^d = \mathbb{R} \times S^d$ may be defined as the cylinder with radius 1 around the time axis in the $d + 1$ -dimensional Minkowski spacetime M^{d+1} , equipped with the induced metric.

Denoting the coordinates in M^{d+1} by (t_E, \mathbf{x}_E, w_E) and the coordinates in M^d by (t_M, \mathbf{x}_M) , we consider the embedding

$$(3.5) \quad \begin{cases} \mathbf{x}_E = (\eta + r_M^2)^{-1/2} \mathbf{x}_M \\ w_E = \text{sgn}(\eta)(\eta + r_M^2)^{-1/2} \\ t_E = \arctan(t_M + r_M) + \arctan(t_M - r_M) \end{cases}$$

which maps the d -dimensional Minkowski space into the Einstein universe, where we have set $r_M = |\mathbf{x}_M|$, $\eta = \frac{1}{2}(1 + t_M^2 - r_M^2)$.

If we now use the cylindrical coordinates (t_E, θ_E, ψ_E) in M^{d+1} to describe E^d , and the cylindrical coordinates (t_M, r_M, θ_M) in M^d , we get

- the metric tensor of E^d is $ds_E^2 = dt_E^2 - d\psi_E^2 - \sin^2 \psi_E d\Omega(\theta_E)^2$, where $d\Omega(\theta_E)^2$ denotes the metric tensor of the $(d-2)$ -dimensional unit sphere;
- the metric tensor for M^d is $ds_M^2 = dt_M^2 - dr_M^2 - r_M^2 d\Omega(\theta_M)^2$;
- the embedding above can be written as

$$(3.6) \quad \begin{cases} \theta_E = \theta_M \\ \psi_E = \arctan(t_M + r_M) - \arctan(t_M - r_M) \\ t_E = \arctan(t_M + r_M) + \arctan(t_M - r_M). \end{cases}$$

A simple calculation shows that the metric tensor on M^d is pulled back to the following metric on E^d :

$$ds^2 = \frac{1}{4} \sec^2 \left(\frac{t_E + \psi_E}{2} \right) \sec^2 \left(\frac{t_E - \psi_E}{2} \right) ds_E^2,$$

showing that the embedding is conformal and that the image of M^d is the “double cone” of E^d given by

$$(3.7) \quad -\pi < t_E \pm \psi_E < \pi.$$

Remark 3.2. In the two dimensional case ψ is the only angle coordinate, hence it ranges from $-\pi$ to π , and the previous inequality is indeed drawn as a double cone. In higher dimension, $\psi \in [0, \pi]$, however the inequality still describes a double cone in E^d with center (t_0, \mathbf{v}_0) :

$$(3.8) \quad \{(t, \mathbf{v}) \in \mathbb{R} \times S^{d-1} : |t - t_0| + d(\mathbf{v}, \mathbf{v}_0) < \pi\},$$

where $t_0 = 0$, \mathbf{v}_0 is the point $\mathbf{x}_E = 0$, $w_E = 1$, and $d(\cdot, \cdot)$ denotes the geodesic distance in E^d .

The conformal Lie algebra of M^d is $o(d, 2)$. If $d \geq 3$, the quotient of the universal covering of $SO_0(d, 2)$ by the stabilizer of a point is E^d . However the action of the universal covering of $SO_0(d, 2)$ is not effective, since there is a \mathbb{Z}_2 component in the center acting trivially on E^d . The corresponding quotient is (the identity component of) the conformal group of E^d .

Indeed let us now consider the map γ in $E^d = \mathbb{R} \times S^{d-1}$ given by $\gamma : (t_E, \mathbf{v}) \mapsto (t_E + \pi, -\mathbf{v})$, where $\mathbf{v} \mapsto -\mathbf{v}$ is the antipodal map. It is easy to see that γ belongs to $\text{Conf}(E^d)$ and the “double cone” above is a fundamental domain for the corresponding action of \mathbb{Z} on E^d . Therefore the quotient is the conformal completion $\overline{M^d}$ of M^d , which is usually called the Dirac-Weyl compactification of $\overline{M^d}$. Since Γ^2 is central $\text{Conf}_0(E^d)$, the quotient $SO_0(d, 2) = \text{Conf}(E^d)/2\mathbb{Z}$ acts on $\overline{M^d}$. If d is even, such action is not effective on $\overline{M^d}$, and the (quasi-global) conformal group of M^d is $PSO_0(d, 2)$. If d is odd, the action is effective, and $\text{Conf}(M^d) = SO_0(d, 2)$.

If $d = 2$, the conformal group is infinite dimensional, however we shall still set $\text{conf}(M^2) = o(2, 2)$. Moreover, E^2 is not simply connected, indeed the procedure outlined above would give as $\widetilde{M^2}$ the universal covering of E^2 . However, E^2 is the only globally hyperbolic covering of $\overline{M^2}$ where the image of M^2 has empty space-like complement. As we shall see below, this condition is necessary in order to lift a conformal net on E^2 to a local net, therefore we shall write $\widetilde{M^d} = E^d$ when $d = 2$ too.

3.1.3. The embedding of dS^d . The de Sitter space dS^d (of radius ρ) may be described in terms of the coordinates (τ, θ_S, ψ_S) , where τ varies in $(0, \pi)$, $(\theta_S$ are

spherical coordinates in S^{d-2} and ψ_S varies in $[0, \pi]$, such that $(\boldsymbol{\theta}_S, \psi_S)$ are spherical coordinates in S^{d-1} . Then the embedding of dS^d in M^{d+1} is

$$(3.9) \quad \begin{cases} t = -\rho \cot \tau \\ \mathbf{x} = \rho \cdot (\sin \tau)^{-1} \mathbf{v}(\boldsymbol{\theta}_S, \psi_S), \end{cases}$$

where $\mathbf{v}(\boldsymbol{\theta}_S, \psi_S)$ denotes a point in S^{d-1} expressed in terms of spherical coordinates. In terms of these coordinates the metric tensor is

$$ds^2 = \frac{\rho^2}{\sin^2 \tau} (d\tau^2 - d\psi_S^2 - \sin^2(\psi_S) d\Omega(\boldsymbol{\theta}_S)^2).$$

Therefore the embedding of dS^d in E^d

$$(3.10) \quad \begin{cases} \varphi_E = \varphi_S \\ \boldsymbol{\theta}_E = \boldsymbol{\theta}_S \\ \psi_E = \psi_S \\ t_E = \tau \end{cases}$$

is conformal and maps dS^d to the “rectangle” of E^d

$$(3.11) \quad \{(t_E, \mathbf{x}_E, w_E) : |\mathbf{x}_E|^2 + w_E^2 = 1, 0 < t_E < \pi\}.$$

Again, the rectangle is a fundamental domain for the action of \mathbb{Z} on E^d induced by Γ . Therefore $\text{Conf}(M^d) = \text{Conf}(dS^d)$ and $\overline{M^d} = \overline{dS^d}$, $\text{Conf}(dS^d)$ acting quasiglobally on dS^d . When $d = 2$ we define $\text{conf}(dS^2) = o(2, 2)$. Let us note that, opposite to the M^2 case, the homogeneous space given by the quotient of $\text{CONF}_{loc}(dS^2)$ by the stabilizer of a point is exactly E^2 , not its covering.

3.1.4. The conformal steady-state universe. The intersection of the conformal images of M^d and dS^d in E^d is the steady-state universe. Composing the previous maps we may therefore obtain a conformal map from the subspace $\{t_M > 0\}$ in the Minkowski space to the steady state subspace of the de Sitter space.

The map can be written as a map from $\{t_M > 0\}$ in M^d to M^{d+1} , with range $\{(t, \mathbf{x}, w) \in M^{d+1} : -t^2 + |\mathbf{x}|^2 + w^2 = \rho^2, w > t\}$:

$$(3.12) \quad \begin{cases} t = -\rho \cdot \frac{t_M^2 - |\mathbf{x}_M|^2 - 1}{2t_M} \\ \mathbf{x} = \rho \cdot \frac{\mathbf{x}_M}{t_M} \\ w = -\rho \cdot \frac{t_M^2 - |\mathbf{x}_M|^2 + 1}{2t_M}. \end{cases}$$

The image of the steady state universe in E^d is not a fundamental domain for some Γ , therefore there is no conformally complete manifold in which it embeds densely.

Let us note for further reference that time translations for $t_M > 0$ are mapped to endomorphisms of the steady-state universe, and that the (incomplete) time-like geodesic $\{\mathbf{x}_M = 0, t_M > 0\}$ is mapped to the de Sitter-complete geodesic

$$(3.13) \quad \begin{cases} t = -\rho \cdot \frac{t_M^2 - 1}{2t_M} \\ \mathbf{x} = 0 \\ w = -\rho \cdot \frac{t_M^2 + 1}{2t_M}. \end{cases}$$

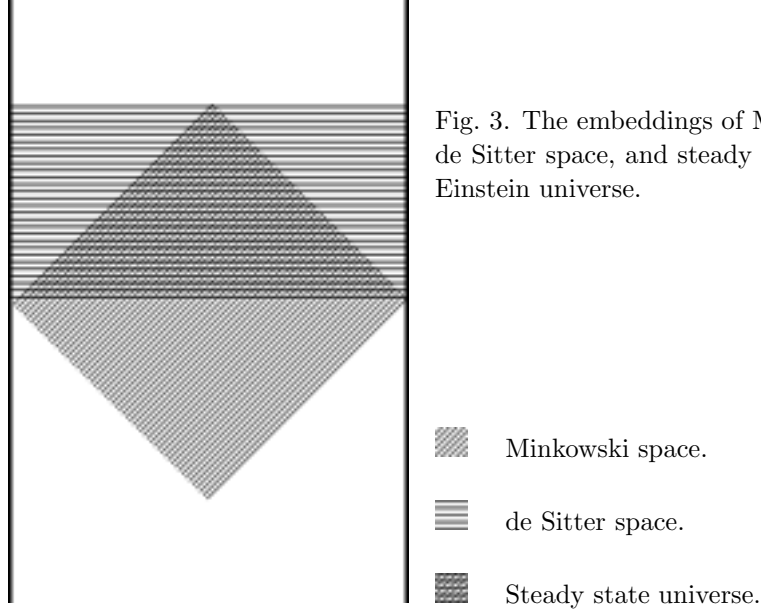


Fig. 3. The embeddings of Minkowski space, de Sitter space, and steady state universe in Einstein universe.

3.2. Conformal nets on de Sitter and Minkowski spacetimes. Let \mathcal{M} be a spacetime on which the local action of the conformal group is transitive.

A net \mathcal{A} of local algebras on \mathcal{M} is *conformal* if there exists a unitary local representations U of $\text{CONF}_{loc}(\mathcal{M})$ acting covariantly: for each fixed double cone \mathcal{O} there exists a neighbourhood \mathcal{U} of the identity in $\text{CONF}_{loc}(\mathcal{M})$ such that $g\mathcal{O} \subset \mathcal{M}$ for all $g \in \mathcal{U}$ and

$$U(g)\mathcal{A}(\mathcal{O})U(g)^{-1} = \mathcal{A}(g\mathcal{O}), \quad \forall g \in \mathcal{U}.$$

Proposition 3.3. *A conformal net \mathcal{A} on \mathcal{M} lifts to a net $\tilde{\mathcal{A}}$ on $\tilde{\mathcal{M}}$ and the local representation U lifts to a true representation \tilde{U} of $\text{CONF}_{loc}(\mathcal{M})$ under which $\tilde{\mathcal{A}}$ is covariant.*

Proof. The results follow as in [10]: for any region $\mathcal{O}_1 \equiv g\mathcal{O} \subset \tilde{\mathcal{M}}$, $\mathcal{O} \subset \mathcal{M}$, $g \in \text{CONF}_{loc}(\mathcal{M})$, we set $\tilde{\mathcal{A}}(g\mathcal{O}) = \tilde{U}(g)\mathcal{A}(\mathcal{O})\tilde{U}(g^{-1})$, and observe that $\mathcal{A}(\mathcal{O}_1)$ is well defined since $\text{CONF}_{loc}(\mathcal{M})$ is simply connected. Then we extend the net $\tilde{\mathcal{A}}$ on all $\tilde{\mathcal{M}}$ by additivity. \square

Let us notice that we did not assume \mathcal{A} to be local, namely that commutativity at spacelike distance is satisfied. In particular we did not prove that $\tilde{\mathcal{A}}$ is local. This was proved in [10] for local nets on the Minkowski space, and the proof easily extends to a spacetime \mathcal{M} where the conformal group acts quasi globally and whose conformal completion is the Dirac-Weyl space, as is the case for the de Sitter space. We shall prove a more general result here.

We say that a (local) unitary representation of $\text{Conf}(E^d)$ has positive energy if the generator of the one-parameter group of time translations on E^d is positive.

Let us denote by \mathcal{K} the set of double cones of E^d (the definition is analogous as in the de Sitter case), and by $\Lambda_{\mathcal{O}}$ the one-parameter group of conformal transformation of E^d , that can be defined by requiring that Λ_W is the boost one parameter group associated with W if W is a wedge of M^d embedded in E^d , and $\Lambda_{\mathcal{O}}(t) = g\Lambda_W(t)g^{-1}$ if $\mathcal{O} \in \mathcal{K}$ and g is a conformal transformation such that $gW = \mathcal{O}$.

We shall say that a net on E^d satisfies the *double cone KMS property* if, for any $\mathcal{O} \in \mathcal{K}$, $(\Omega, \cdot \Omega)$ is a KMS state on the algebra associated with \mathcal{O} w.r.t. the evolution implemented by $U(\Lambda_{\mathcal{O}}(\cdot))$.

Theorem 3.4. *Let \mathcal{M} be a spacetime s.t. $\widetilde{\mathcal{M}} = E^d$. Then a local conformal net \mathcal{A} on \mathcal{M} with positive energy lifts to a local net $\tilde{\mathcal{A}}$ on E^d which is covariant under the (orientation-preserving) conformal group $\text{CONF}_+(E^d)$.*

$\tilde{\mathcal{A}}$ satisfies Haag duality and the double cone KMS property.

Proof. By the above proposition we get a net on $\widetilde{\mathcal{M}}$ which is covariant under the universal covering of $SO(d, 2)$. Then modular unitaries associated with double cones act geometrically, as in [10] Lemma 2.1. Now we fix two causally disjoint double cones $\mathcal{O}, \mathcal{O}_1 \subset \mathcal{M}$. Then if $\Lambda_{\mathcal{O}}$ is the one parameter group of conformal transformations corresponding to the (rescaled) modular group of $\mathcal{A}(\mathcal{O})$, we have that $\mathcal{A}(\mathcal{O})$ commutes with $\tilde{\mathcal{A}}(\Lambda_{\mathcal{O}}(t)\mathcal{O}_1)$ for any t . Let us assume for the moment that $d > 2$. Then $\Lambda_{\mathcal{O}}$ leaves globally invariant the spacelike complement \mathcal{O}' of \mathcal{O} in $\widetilde{\mathcal{M}}$, indeed its action is implemented by the modular group of $\tilde{\mathcal{A}}(\mathcal{O}')$ at $-t$. Therefore the algebra

$$\bigvee_{t \in \mathbb{R}} \tilde{\mathcal{A}}(\Lambda_{\mathcal{O}}(t)\mathcal{O}_1)$$

is a subalgebra of $\tilde{\mathcal{A}}(\mathcal{O}')$, is globally stable under the action of $\Delta_{\mathcal{O}'}^{it}$, commutes with $\mathcal{A}(\mathcal{O})$ and is cyclic for the vacuum. By the Takesaki theorem the subalgebra indeed coincides with $\tilde{\mathcal{A}}(\mathcal{O}')$. This implies that the net is local by covariance. One then proves the geometric action of J , thus extending the representation to conformal transformations which does not preserve the time orientation. \tilde{U} is a representation of the conformal group of E^d rather than of its simply connected two-covering by a spin and statistics argument (cf. [25, 38, 27]). The last properties are proved as in [10]. In the low-dimensional case we may assume that \mathcal{O}_1 is “on the right” of \mathcal{O} and $\cup_{t \in \mathbb{R}} \Lambda_{\mathcal{O}}(t)\mathcal{O}_1 = \mathcal{O}^R$, where \mathcal{O}^R is the closest smallest region on the right of \mathcal{O} which is globally invariant under $\Lambda_{\mathcal{O}}(t)$. As before, we prove that $\mathcal{A}(\mathcal{O}^R) = \mathcal{A}(\mathcal{O})'$. Now there exists a suitable conformal rotation whose lift $R(t)$ to E^d satisfies $R(\pi)\mathcal{O} = \mathcal{O}^R$, $R(\pi)\mathcal{O}^R = (\mathcal{O}^R)^R$, and so on. Therefore, $\mathcal{A}(R(2\pi)\mathcal{O}) = \mathcal{A}(\mathcal{O})'' = \mathcal{A}(\mathcal{O})$, namely the net actually lives on E^d . The rest of the proof goes on as before. \square

Remark 3.5. In the proof above, we proved in particular that, when $d \leq 2$, the extension of the net satisfying locality necessarily lives on E^d , and not on its universal covering.

Besides the Minkowski space and the de Sitter space, Th. 3.4 applies to the Robertson-Walker space RW^d , to the Rindler wedge and many others. In particular, there is a bijection between isomorphism classes:

$$\boxed{\text{local conformal nets on } M^d \rightleftharpoons \text{local conformal nets on } dS^d}$$

In the following theorem we describe what the positive energy condition on M^d becomes on dS^d under this correspondence.

Theorem 3.6. *There is a natural correspondence between isomorphism classes of*

- (i) *Local conformal nets on M^d with positive energy;*
- (ii) *Local conformal nets on dS^d with the KMS property for geodesic observers;*
- (iii) *Local conformal nets on E^d with positive energy.*

Here positive energy on $E^d = \mathbb{R} \times S^{d-1}$ means that the one-parameter group of time translations (on \mathbb{R}) is implemented by a unitary group with positive generator.

Proof. (i) \Leftrightarrow (iii): Let us note that the universal covering $\widetilde{SO}(d, 2)$ of $SO_0(d, 2)$ is also the universal covering of $SL(2, \mathbb{R})$. Since the covariance unitary representation U of $\widetilde{SO}(d, 2)$ is the same, it suffices to show that the two one-parameter unitary subgroups of U in question both have or not have positive generators.

Let's consider the group generated by time translation, dilations and ray inversion in M^d . This group is isomorphic to $PSL(2, \mathbb{R})$ and acts on time axis of M^d . U restricts to a unitary representation of $PSL(2, \mathbb{R})$ thus, by a well known fact (see e.g. [37], positivity of time translations on M^d is equivalent to positivity of conformal rotations (the generator corresponding to the rotation subgroup of $PSL(2, \mathbb{R})$ is positive). Now the above rotation group provides the time translations on E^d , hence the positivity of the corresponding one-parameter subgroup of $\widetilde{SO}(d, 2)$ is a consequence of the mentioned equivalence of positive energy conditions for unitary representations of (the universal covering of) $PSL(2, \mathbb{R})$.

(iii) \Rightarrow (ii) has been proved above.

(ii) \Leftrightarrow (i): It is known that a Poincaré covariant net for which the vacuum is KMS for the algebra of a wedge acted upon by the boosts satisfies the spectrum condition, see e.g. [16, 11]. Since geodesic KMS property for a net on dS^d is equivalent to the KMS property for any wedge for the corresponding net on M^d , we get the thesis. \square

If \mathcal{O} is a double cone with vertices x and y , call B a basis of \mathcal{O} if B is the part of a Cauchy surface contained in \mathcal{O} and the closure of B contains the points lightlike to both x and y . We shall say that a net \mathcal{A} satisfies the *local time-slice property* if for any double cone \mathcal{O} if

$$\mathcal{A}(\underline{\mathcal{Q}}) = \mathcal{A}(\mathcal{O})$$

with $\mathcal{O} \in \mathcal{K}$ and $\underline{\mathcal{Q}} \subset \mathcal{O}$ an open slice around a basis B of \mathcal{O} , namely a tubular neighbourhood of B contained in \mathcal{O} (thus $\mathcal{O} = \underline{\mathcal{Q}}''$). Note that, by an iteration/compactness argument, for the local timelike slice property to hold it is enough to assume $\mathcal{A}(\underline{\mathcal{Q}}) = \mathcal{A}(\mathcal{O})$ where $\underline{\mathcal{Q}}$ is obtained by \mathcal{O} by removing arbitrarily small neighbourhoods of the vertices of \mathcal{O} (by using additivity).

Corollary 3.7. *Let \mathcal{A} be a conformal net on dS^d . \mathcal{A} is Haag dual iff it satisfies the local time-slice property.*

Proof. With dS^d is embedded in E^d as above and \mathcal{O} a double cone in dS^d , we have $\mathcal{A}(\mathcal{O})' = \mathcal{A}(\mathcal{O}'_E)$ where \mathcal{O}'_E is the causal complement of \mathcal{O} in E^d . Thus \mathcal{A} is Haag dual on dS^d iff $\mathcal{A}(\mathcal{O}') = \mathcal{A}(\mathcal{O}'_E)$, where $\mathcal{O}' = \mathcal{O}'_E \cap dS^d$ is the causal complement of \mathcal{O} in dS^d . Now \mathcal{O}'_E is a double cone in E^d and \mathcal{O}' is a timelike slice for \mathcal{O}' , so Haag duality on dS^d is satisfied iff the time-slice holds for \mathcal{O}'_E . We can now map, by a conformal transformation, \mathcal{O}'_E to any other double cone contained in dS^d , thus the time-slice property holds on dS^d iff it holds for \mathcal{O}'_E . \square

Thus, under a general assumption (local time-slice property), all conformal nets on dS^2 are Haag dual. One should compare this with the Minkowski spacetime case, where Haag duality for conformal nets is equivalent to a strong additivity requirement: removing a point from the basis B of \mathcal{O} we have $\mathcal{A}(\mathcal{O}) = \mathcal{A}(B \setminus \{\text{pt}\})''$ [33]. As a consequence, if two conformal nets on M^d and dS^d are conformally related as above, then

$$\boxed{\text{Haag duality on } M^d \implies \text{Haag duality on } dS^d}$$

but the converse is not true.

3.3. Modular covariance and the maximal conformal subnet. We have shown that on spacetimes that can be conformally embedded in E^d , a local, locally conformal net can be lifted to a local, globally conformal net on E^d with the double cone KMS property, namely to a net for which the modular groups of double cones have a geometric action. Indeed a converse is true. Assume we have a spacetime \mathcal{M} such that $\widetilde{\mathcal{M}} = E^d$. We shall say that a subregion \mathcal{O} of \mathcal{M} is a

double cone if it can be conformally identified with a double cone in E^d . Given a net \mathcal{A} on the double cones of \mathcal{M} acting on a Hilbert space with a given vector Ω , such that, for any double cone $\mathcal{O} \subseteq \mathcal{M}$, Ω is cyclic and separating for $\mathcal{A}(\mathcal{O})$, we shall consider the following property for the algebra $\mathcal{A}(\mathcal{O})$:

- *Local modular covariance*: for every double cone $\tilde{\mathcal{O}} \subset \mathcal{O}$, we have

$$\Delta_{\mathcal{O}}^{it} \mathcal{A}(\tilde{\mathcal{O}}) \Delta_{\mathcal{O}}^{it} = \mathcal{A}(\Lambda_{\mathcal{O}}(-2\pi t)\tilde{\mathcal{O}}).$$

Local modular covariance was introduced in [23] under the name of weak modular covariance, where it was proven that weak modular covariance for wedges plus essential duality is equivalent to modular covariance, hence reconstructs the Poincaré covariant representation, for nets on the Minkowski space.

Theorem 3.8. *Let \mathcal{O}_0 be a spacetime which can be conformally identified with a double cone in E^d , and assume we are given a net $\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O})$ of local algebras, $\mathcal{O} \subset \mathcal{O}_0$, acting on a Hilbert space with a given vector Ω , such that, for any double cone $\mathcal{O} \subseteq \mathcal{O}_0$, Ω is cyclic and separating for $\mathcal{A}(\mathcal{O})$ and the local modular covariance property holds. Then the net extends to a conformal net on (the universal covering of) E^d . If \mathcal{A} is local, then the extended net is indeed a local conformal net on E^d .*

The proof requires some steps.

We first construct “half-sided modular translations”. Let us identify \mathcal{O}_0 with a future cone in M^d , and denote by $v \mapsto \tau^+(v)$ the subgroup of the conformal group isomorphic to \mathbb{R}^d consisting of M^d translations, in such a way that when v is a causal future-pointing vector $\tau^+(v)$ implements endomorphisms of \mathcal{O}_0 . These transformations can be seen as conformal translations which fix the upper vertex of \mathcal{O}_0 . In the same way we get a family $v \mapsto \tau^-(v)$ of conformal translations fixing the lower vertex of \mathcal{O}_0 , and such that $\tau^-(v)$ implements endomorphisms of \mathcal{O}_0 when v is a causal past-pointing vector.

For any causal future-pointing vector v , we may implement the translation $t \mapsto \tau^+(tv)$ by a one-parameter unitary group $T^+(tv)$ with positive generator *la Wiesbrock*. Borchers relations are satisfied: $\Delta_{\mathcal{O}_0}^{it} T^+(v) \Delta_{\mathcal{O}_0}^{-it} = T^+(e^{-2\pi t}v)$.

Translations $T^-(v)$, for causal (past-pointing) vector v , are constructed analogously.

Lemma 3.9. *The T^\pm translations associated with \mathcal{O}_0 act geometrically on subregions, whenever it makes sense:*

$$(3.14) \quad \text{Ad } T^\pm(v) \mathcal{A}(\mathcal{O}) = \mathcal{A}(\tau^\pm(v)\mathcal{O}), \quad \text{if } \tau^\pm(v)\mathcal{O} \subset \mathcal{O}_0.$$

Proof. First assume \mathcal{O} is compactly contained in \mathcal{O}_0 and the translation “goes inside”, namely it is of the form $\tau^+(v)$ with v a causal future-pointing vector or $\tau^-(v)$ with v a causal past-pointing vector. Then there exists an $\varepsilon > 0$ such that $\tau^+(\varepsilon v)\mathcal{O} \subset \mathcal{O}_0$. Therefore, by Borchers relations, $T^+(\varepsilon(e^{-2\pi t} - 1)v) = \Delta_{\mathcal{O}_0}^{it} \Delta_{\tau^+(\varepsilon v)\mathcal{O}_0}^{-it}$, and the thesis follows.

Then, by additivity, one can remove the hypothesis that \mathcal{O} is compactly contained in \mathcal{O}_0 . Indeed, by local modular covariance, for any $\mathcal{O} \subset \mathcal{O}_0$, the von Neumann algebra generated by the local algebras associated with compactly contained subregions of \mathcal{O} is globally stable for $\Delta_{\mathcal{O}}^{it}$, therefore, by Takesaki Theorem, it coincides with $\mathcal{A}(\mathcal{O})$.

We have proved that (3.14) holds for $\tau^+(v)$ whenever v is a causal future-pointing vector, hence, applying $\text{Ad } T^+(-v)$, one gets $\text{Ad } T^+(-v) \mathcal{A}(\mathcal{O}) = \mathcal{A}(\tau^+(-v)\mathcal{O})$ whenever $\tau^+(-v)\mathcal{O} \subset \mathcal{O}_0$. The thesis follows. \square

Lemma 3.10. *T^+ is indeed a representation of \mathbb{R}^d , and the same holds for T^- . They act geometrically on subregions, whenever it makes sense.*

Proof. First we prove, as in [23], that $[T^+(v), T^+(w)] = 0$. By the previous point, the multiplicative commutator

$$(3.15) \quad c(s, t) := T^+(-sv)T^+(-tw)T^+(sv)T^+(tw)$$

has a geometric action, hence stabilizes, the algebras $\mathcal{A}(\mathcal{O})$, for $s, t \geq 0$. Therefore it commutes with $\Delta_{\mathcal{O}}^{it}$ and with the translations themselves. With simple manipulations we get $c(s, t) = c(-s, -t) = c(-s, t)^* = c(s, -t)^*$, namely $c(s, t)$ commutes with translations for any s, t , hence $T^+(sv), T^+(tw)$ generate a central extension of \mathbb{R}^2 . By positivity of the generators the commutator has to vanish.

In an analogous way one shows that $c(t) := T^+(-t(v+w))T^+(tv)T^+(tw)$ is central, hence is a one-parameter group, and by Borchers relations $c(\lambda t) = c(t)$ for any positive λ , namely $c(t) = 1$. The relations for T^- and the geometric action follows as before. \square

Now we construct the group \mathcal{G} . For any $\mathcal{O} \subseteq \mathcal{O}_0$, define $\mathcal{G}(\mathcal{O})$ as the group generated by

$$\{\Delta_{\mathcal{O}}^{it} : \tilde{\mathcal{O}} \subseteq \mathcal{O}\}.$$

Lemma 3.11. *$\mathcal{G}(\mathcal{O})$ is independent of \mathcal{O} .*

Proof. Let us note that $\mathcal{G}(\tau^\pm(v)\mathcal{O})$ is a subgroup of $\mathcal{G}(\mathcal{O})$ and clearly contains $T^\pm(v)$, hence coincides with $\mathcal{G}(\mathcal{O})$. Repeating this argument we get that $\mathcal{G}(\mathcal{O})$ does not depend on \mathcal{O} . \square

We shall denote this group simply by \mathcal{G} . Let us note that \mathcal{G} is generated by a finite number of one-parameter groups: setting $\mathcal{O}_k = \tau^+(v_k)\mathcal{O}_0$, $k = 1, \dots, d$, $\mathcal{O}_{k+d} = \tau^-(v_k)\mathcal{O}_0$, $k = 1, \dots, d$, the one-parameter groups $\Delta_{\mathcal{O}_k}^{it}$, $k = 0, \dots, 2d$ generates all translations $T^\pm(v)$, hence \mathcal{G} by covariance.

Then we construct the central extension. The one-parameter groups Λ_k $k = 1, \dots, d$, generate the conformal group $SO_0(d, 2)$. Pick $(d+1)(d+2)/2$ functions $g_i(t)$ with values in $SO_0(d, 2)$, each given by a product of Λ_k 's, such that the Lie algebra elements $g'_i(0)$ form a basis for $so(d, 2)$. Since $(d+1)(d+2)/2 \geq 2d+1$ one may assume that $g_i(t) = \Lambda_{\mathcal{O}_{k-1}}(-2\pi t)$, $i = 1, \dots, 2d+1$. Then the map $F(\mathbf{t}) = g_1(t_1) \cdots g_n(t_n)$ $n = (d+1)(d+2)/2$, is a local diffeomorphism from \mathbb{R}^n to the conformal group. Now use the identification $\Lambda_{\mathcal{O}}(-2\pi t) \leftrightarrow \Delta_{\mathcal{O}}^{it}$ to get a map G from \mathbb{R}^n to the group $\mathcal{G} = \mathcal{G}(\mathcal{O}_0 \dots \mathcal{O}_{2d})$ generated by the $\Delta_{\mathcal{O}_k}^{it}$, $k = 0, \dots, 2d$, and finally obtain a map $H = G \cdot F^{-1}$ from a neighborhood \mathcal{V} of the identity in $SO_0(d, 2)$, to \mathcal{G} . Observe that $\text{Ad } H(g)\mathcal{A}(\mathcal{O}) = \mathcal{A}(g\mathcal{O})$ whenever each step makes sense.

Lemma 3.12. *The inverse of the map H gives rise to a homomorphism from \mathcal{G} to $SO_0(d, 2)$ which is indeed a central extension.*

Proof. First we show that H is a local homomorphism to \mathcal{G}/\mathcal{Z} , \mathcal{Z} denoting the center of \mathcal{G} .

Choose a region $\tilde{\mathcal{O}}$ compactly contained in \mathcal{O}_0 . Now, possibly restricting \mathcal{V} , one may assume that $g\tilde{\mathcal{O}} \subset \mathcal{O}$ for any $g \in \mathcal{V}$. As a consequence, if $g, h, gh \in \mathcal{V}$, then $H(gh)^*H(g)H(h)$ implements an automorphism of $\mathcal{A}(\mathcal{O})$, for $\mathcal{O} \subset \tilde{\mathcal{O}}$, namely commutes with the corresponding modular groups, hence is in the center of \mathcal{G} .

Now we extend the map H to a homomorphism from the universal covering $\widetilde{SO}(d, 2)$ of $SO_0(d, 2)$ to \mathcal{G}/\mathcal{Z} , and observe that since all normal subgroups of $\widetilde{SO}(d, 2)$ are central, we get an isomorphism from a suitable covering $\tilde{\mathcal{C}}$ of $SO_0(d, 2)$ to \mathcal{G}/\mathcal{Z} . The inverse gives rise to a homomorphism from \mathcal{G} to $SO_0(d, 2)$ which is indeed a central extension. \square

Proof of Theorem 3.8. The arguments in [11] show that the extension is weak Lie type, hence gives rise to a strongly continuous representation U of $\widetilde{SO}(d, 2)$. Such

representation acts geometrically on the algebras $\mathcal{A}(\mathcal{O})$ whenever it makes sense, therefore, by Proposition 3.3 we get a CFT on (the universal covering of) E^d . If \mathcal{A} is local, the extension is indeed a local net on dS^d by Theorem 3.4. \square

In the following Corollary we characterize conformal theories in terms of local modular covariance.

Corollary 3.13. *Let \mathcal{M} be a spacetime for which $\widetilde{\mathcal{M}} = E^d$. Then there is a natural correspondence between*

- *Local conformal nets on E^d with positive energy;*
- *Local nets on \mathcal{M} with local modular covariance for double cones.*

Proof. Assume we are given a local net \mathcal{A} on \mathcal{M} satisfying local modular covariance for double cones. For any double cone $\mathcal{O} \subset \mathcal{M}$, Theorem 3.8 gives a local conformal net $\widetilde{\mathcal{A}}_{\mathcal{O}}$ on E^d , based on the restriction of \mathcal{A} to \mathcal{O} .

Now embed \mathcal{M} in E^d , and observe that, by Lemma 3.11, if $\mathcal{O}_1 \subset \mathcal{O}_2 \subset dS^d$ the two nets $\widetilde{\mathcal{A}}_{\mathcal{O}_i}$, $i = 1, 2$, on E^d coincide. From this we easily get that all nets $\widetilde{\mathcal{A}}_{\mathcal{O}}$ based on $\mathcal{A}|_{\mathcal{O}}$ coincide, hence their restriction to \mathcal{M} coincides with \mathcal{A} . The converse implication follows by Theorem 3.4. \square

Note that the unitary representation of the conformal group is unique [10] because it is generated by the unitary modular groups associated with double cones (cf. Thm. 3.8).

A further consequence of Corollary 3.13 is *additivity* for a conformal net \mathcal{A} : if $\mathcal{O}, \mathcal{O}_i$ are double cones and $\mathcal{O} \subset \cup_i \mathcal{O}_i$, then $\mathcal{A}(\mathcal{O}) \subset \vee_i \mathcal{A}(\mathcal{O}_i)$. This can be proved by the argument in [21].

We now return to the de Sitter spacetime dS^d , with any dimension d . Let \mathcal{A} be a local net on dS^d and \mathcal{B} be a subnet of \mathcal{A} . We shall say that \mathcal{B} is a *conformal subnet* if its restriction \mathcal{B}_0 to $\mathcal{H}_{\mathcal{B}}$ is a conformal net. Now, given any local net \mathcal{A} and $\mathcal{O} \in \mathcal{K}$, we set

$$(3.16) \quad \mathcal{C}(\mathcal{O}) = \{X \in \mathcal{A}(\mathcal{O}) : \Delta_{\mathcal{O}_0}^{it} X \Delta_{\mathcal{O}_0}^{-it} \in \mathcal{A}(\Lambda_{\mathcal{O}_0}(-2\pi\rho t)\mathcal{O}), \forall \mathcal{O}_0 \in \mathcal{K}, \mathcal{O}_0 \supset \mathcal{O}\}.$$

It is immediate to check that $\mathcal{C}(\mathcal{O})$ is a von Neumann subalgebra of $\mathcal{A}(\mathcal{O})$. Moreover \mathcal{C} is covariant w.r.t. the unitary representation of $SO_0(d, 1)$ because if $X \in \mathcal{C}(\mathcal{O})$, then $\text{Ad } U(g)X \in \mathcal{A}(g\mathcal{O})$ and, for any $\mathcal{O}_0 \supset g\mathcal{O}$,

$$(3.17) \quad \begin{aligned} \text{Ad } \Delta_{\mathcal{O}_0}^{it} U(g)X &= \text{Ad } U(g) \Delta_{g^{-1}\mathcal{O}_0}^{it} X \\ &\in \text{Ad } U(g) \mathcal{A}(\Lambda_{g^{-1}\mathcal{O}_0}(-2\pi\rho t)\mathcal{O}) = \mathcal{A}(\Lambda_{\mathcal{O}_0}(-2\pi\rho t)g\mathcal{O}), \end{aligned}$$

namely $U(g)XU(g)^{-1} \in \mathcal{C}(g\mathcal{O})$.

Finally \mathcal{C} is isotonic, thus \mathcal{C} is a subnet of \mathcal{A} .

Theorem 3.14. *A local net \mathcal{A} on dS^d has a unique maximal conformal expected subnet \mathcal{C} . It is given by eq. (3.16).*

Proof. Let \mathcal{B} be a conformal expected subnet of \mathcal{A} . Then \mathcal{B}_0 is weakly additive by Lemma 2.5, hence satisfies the Reeh-Schlieder property by Lemma 2.3. So the projection E onto $\mathcal{H}_{\mathcal{B}}$ implements all the expectations $\varepsilon_{\mathcal{O}}$ and commutes with all $\Delta_{\mathcal{O}}$ by Takesaki theorem. By Corollary 3.13, local modular covariance is satisfied for $\mathcal{B}(\mathcal{O})$, hence if X is an element of the algebra $\mathcal{B}(\mathcal{O})$ it belongs to $\mathcal{C}(\mathcal{O})$.

Thus we have only to show that the subnet \mathcal{C} is conformal and expected. Clearly $\Delta_{\mathcal{O}}^{it} \mathcal{C}(\mathcal{O}) \Delta_{\mathcal{O}}^{-it} = \mathcal{C}(\mathcal{O})$, thus \mathcal{C} is expected by Takesaki theorem. Also, by construction, local modular covariance holds true, so \mathcal{C} is conformal by Corollary 3.13. \square

4. THE DETHERMALIZATION EFFECT

In the flat Minkowski spacetime, the world line of an inertial particle is a causal line. The corresponding evolution on a quantum field in the vacuum state is implemented by a one-parameter translation unitary group whose infinitesimal generator, “energy”, is positive. A uniformly accelerated observer feels a thermalization (Unruh effect): its orbit is the orbit of a one-parameter group of pure Lorentz transformation that, on the quantum field, is implemented by a one-parameter automorphism group of the von Neumann algebra of the corresponding wedge that satisfies the KMS thermal equilibrium condition at Hawking temperature.

On the other hand, if we consider an inertial observer on the de Sitter spacetime, its world line is the orbit of a boost and it is already thermalized, in the vacuum quantum field state, at Gibbons-Hawking temperature. Our aim is to seek for a different evolution on the de Sitter spacetime with respect to which the vacuum is dethermalized, namely becomes a ground state, an effect opposite to the Unruh thermalization.

4.1. General evolutions. Let us recall a recent proposal of (quasi-)covariant dynamics for not necessarily inertial observers [15]. Our presentation, though strictly paralleling the one in [15], will differ in some respects. Our description is in fact strictly local, therefore local conformal transformations will play the central role. The dynamics will consist of propagators describing the time evolution as seen by the observer, the main requirement being that the rest frame for the observer is irrotational.

Let us consider a (not necessarily parametrized by proper time) observer in a given spacetime \mathcal{M} , namely a timelike, future pointing C^1 curve $\gamma : t \in (-a, a) \mapsto \gamma_t \in \mathcal{M}$.

Then we look for a local evolution for the observer γ , namely a family of maps λ_t from \mathcal{M} to \mathcal{M} such that satisfy the following physical requirements

- $\lambda_t \gamma_0 = \gamma_t$, $t \in (-a, a)$.
- Given $x_0 \in \mathcal{M}$, for each y_0 in some neighborhood of x_0 , the events $\lambda_t(y_0)$, $t \in (-a, a)$, describe, potentially, the worldline of some material particle. This worldline is either disjoint from the observer’s worldline or coincides with it.
- For a suitable y_0 spacelike to x_0 , the axis of a gyroscope carried by the observer at the space-time point $\lambda_t(x_0)$ points towards the point $\lambda_t(y_0)$ at all times t .

As observed in [15], the previous conditions only depend on the conformal structure of the manifold. Therefore we will specify λ_t to be a local conformal transformation of \mathcal{M} , or, more precisely, λ to be a curve in $\text{Conf}(\overline{\mathcal{M}})$.

In this way the notion of local evolution only depends on the conformal class, namely if two metrics belong to the same conformal class they give rise to the same notion of local evolution. From the mathematical point of view, the above requests mean that the range of γ is an orbit of λ , and that for any $t \in (-a, a)$, $(\lambda_t)_*$ (the differential of the transformation $\lambda_t : \mathcal{M} \mapsto \mathcal{M}$) maps orthogonal frames in $T_{\gamma_0} \mathcal{M}$ to orthogonal frames in $T_{\gamma_t} \mathcal{M}$, in such a way that a tangent vector to the curve γ at $t = 0$ is mapped to a tangent vector to the curve γ at the point t , and that every orthogonal vector v to γ at $t = 0$ evolves without rotating to vectors orthogonal to γ , as we will explain.

If we now fix a metric g in the conformal class, we can choose the proper time parametrization, and then look for a curve $\lambda \in \text{Iso}(\mathcal{M}, g)$, the isometry group of \mathcal{M} , namely for a local isometric evolution on γ .

In this way orthonormal frames evolve to orthonormal frames. Recalling the notion of Fermi-Walker transport (cf. [49]), we may reformulate the conditions for a local isometric evolution on γ as follows:

- E_1 : λ is a curve in $\text{Iso}(\mathcal{M}, g)$.
- E_2 : $\lambda_t \gamma_0 = \gamma_t$, $t \in (-a, a)$.
- E_3 : $(\lambda_t)_*$ is the Fermi-Walker transport along the curve γ .

Clearly a local isometric evolution on γ does not exist in general however, if it exists, it is unique.

Proposition 4.1. *Assume λ_t, λ'_t , $t \in (-a, a)$, satisfy properties E_1, E_2, E_3 for a given observer γ . Then λ_t coincides with λ'_t on a suitable neighborhood of γ_0 .*

Proof. By assumption, $\lambda_t^{-1} \cdot \lambda'_t$ is a local isometry fixing the point x_0 whose differential is the identity on $T_{x_0}\mathcal{M}$. Then $\lambda_t^{-1} \cdot \lambda'_t$ acts identically on any geodesic at x_0 , hence coincides with the identity on the injectivity radius neighborhood. \square

Concerning the existence problem, let us first consider a geodesic observer. In this case the Fermi-Walker transport coincides with the parallel transport. Let us recall that a (semi-) Riemannian manifold is *symmetric* if for any $p \in \mathcal{M}$ there exists an involutive isometry σ_p such that p is an isolated fixed point. It is easy to see that de Sitter, Minkowski, and Einstein spacetimes are symmetric.

Proposition 4.2.

- (i) *If γ is a geodesic observer, a local isometric evolution is indeed a one-parameter group of isometries.*
- (ii) *If \mathcal{M} is symmetric, a local isometric evolution exists for any geodesic.*

Proof. (i) First we show that a local isometric evolution for a geodesic γ satisfies $\lambda_t \cdot \lambda_t = \lambda_{2t}$. Indeed, since λ_t is an isometry, $\lambda_t \gamma_s$, $0 \leq s \leq t$ describes a geodesic, and since $(\lambda_t)_* \gamma'_0 = \gamma'_t$, it describes the geodesic γ_{t+s} , $0 \leq s \leq t$. As a consequence, λ_{2t} implements the parallel transport on γ from $T_{\gamma_0}\mathcal{M}$ to $T_{\gamma_t}\mathcal{M}$. By the uniqueness proved in Proposition 4.1 we get the statement.

Now we observe that the previous property implies $\lambda_t \cdot \lambda_s = \lambda_{t+s}$ whenever s/t is rational, hence, by continuity, for any s and t .

(ii) Since γ is geodesic, the Fermi-Walker transport coincides with the parallel transport (cf. [49]). On a symmetric manifold, the existence of isometries implementing the parallel transport is a known fact, see e.g. [3], Thm 8.7. \square

We now study the case of a generic observer. Assume λ is a C^1 one-parameter family of local diffeomorphisms of \mathcal{M} and denote by L_t the vector field given by $L_t(\lambda_t(x)) = \frac{d}{ds}\lambda_s(x)|_{s=t}$. Assume then that the x -derivatives of $L_t(x)$ are jointly continuous, namely that the map

$$(4.1) \quad (t, x, v) \in \mathbb{R} \times T\mathcal{M} \mapsto (\nabla_v L_t)(x)$$

is continuous.

Lemma 4.3. *Let γ be an orbit of λ : $\lambda_t \gamma_0 = \gamma_t$, and let X be a λ -invariant vector field on γ : $(\lambda_t)_* X(\gamma_0) = X(\gamma_t)$. Then, at the point γ_t , the covariant derivative of X on the curve γ satisfies*

$$(4.2) \quad \nabla_{\gamma^*} X = \nabla_X L_t.$$

Proof. Since X is invariant under λ , the commutator $[X, L_t]$ vanishes at the point γ_t . This fact can be proved via a simple computation, where two derivatives should be exchanged. Condition (4.1) ensures that Schwartz Lemma applies.

Then the symmetry of the Levi-Civita connection implies

$$\nabla_{L_t} X = \nabla_X L_t$$

at the point γ_t . Since by definition $L_t(\gamma_t) = \frac{d}{ds}\lambda_s(\gamma_0)|_{s=t} = \frac{d}{ds}\gamma_s|_{s=t}$, we get the thesis. \square

The existence of a local isometric evolution for any observer has been proved in [15] for the de Sitter metric. Property (iii) of the following theorem gives as an extension of this fact.

Theorem 4.4. *Let γ be an observer in \mathcal{M} . The following hold:*

- (i) *There exists a local isometric evolution λ on γ satisfying condition (4.1) iff for every t , γ'_t extends locally to a Killing vector field L_t satisfying (4.1) and $(\nabla_v L_t(\gamma_t), w) = 0$ for every vectors v, w in the rest space of γ_t .*
- (ii) *The existence of a local isometric evolution for any geodesic observer is equivalent to the existence of a local isometric evolution for any observer.*
- (iii) *If \mathcal{M} is symmetric, a local isometric evolution exists for every observer.*

Proof. (i) A curve λ in $\text{Iso}(\mathcal{M})$ satisfying E_2 on γ gives rise, by derivation, to a one-parameter family of Killing vector fields L_t defined by: $L_t(\lambda_t(x)) = \frac{d}{ds}\lambda_s(x)|_{s=t}$. Clearly L_t satisfies $L_t(\gamma_t) = \gamma'_t$.

Conversely a curve L_t of Killing vector fields verifying $L_t(\gamma_t) = \gamma'_t$ gives rise to a curve of local isometries via the equations

$$\begin{aligned} \lambda_0(x) &= x \\ \left. \frac{d\lambda_s(x)}{ds} \right|_{s=t} &= L_t(\lambda_t(x)). \end{aligned}$$

Clearly $\frac{d}{ds}\lambda_s(\gamma_t)|_{s=t} = L_t(\gamma_t) = \gamma'_t$, hence $\lambda_t(\gamma_0) = \gamma_t$, namely condition E_2 .

By condition E_2 , $(\lambda_t)_*$ maps vectors tangent to γ to vectors tangent to γ , hence, being isometric, preserves the rest frame for γ . Therefore it implements the Fermi-Walker transport if and only if tangent rest vectors evolve irrotationally, namely iff the Fermi derivative $\mathbf{F}_{\gamma^*} X = 0$ on γ_t for any λ -invariant vector field X in the rest space of γ . According to [49], Prop. 2.2.1, if P denotes the projection on the rest space, this is equivalent to $P\nabla_{\gamma^*} X = 0$. By equation (4.2), this means that

$$(4.3) \quad P\nabla_v L_t(\gamma_t) = 0, \quad v \in PT_{\gamma_t}\mathcal{M}, \quad \forall t,$$

which is our thesis.

(ii) Assume the existence of a local isometric evolution for any geodesic observer. By Proposition 4.2, L_t does not depend on t , hence condition (4.1) is trivially satisfied. Then, reasoning as in (i) and taking into account that the Fermi-derivative for a geodesic observer is indeed the Levi-Civita connection, we get $\nabla_w L(x) = 0$ for any x in the geodesic, $w \in T_x\mathcal{M}$. Namely, the existence of a local isometric evolution for any geodesic observer is equivalent to the following: for any $(x, v) \in T\mathcal{M}$, there exists a vector field $H = H_{x,v}$ defined in a neighborhood \mathcal{U} of x , such that, if γ is the geodesic determined by (x, v) , H satisfies

$$\begin{aligned} (V, \nabla_H W)(x) &= (\nabla_H V, W)(x), \quad x \in \mathcal{U}, \\ \nabla_w L(x) &= 0, \quad w \in T_x\mathcal{M}, \quad x = \gamma(s), \quad |s| < \varepsilon, \\ L(\gamma_s) &= \gamma'_s, \quad |s| < \varepsilon \end{aligned}$$

where $\gamma(s) \subset \mathcal{U}$ for any $|s| < \varepsilon$. Since any such $H_{x,v}$ would determine a local isometric evolution for γ , Proposition 4.1 imply uniqueness. Hence the existence of a local isometric evolution for any geodesic observer is equivalent to the existence and uniqueness of a local solution for the system above. Let us remember that the solutions of the first equation (the Killing equation) form a finite dimensional space \mathcal{V} , therefore existence and uniqueness can be reformulated as the existence and

uniqueness for the finite-dimensional linear system given by the last two equations, with $L \in \mathcal{V}$. Clearly, both the linear operator and the coefficients depend smoothly on (x, v) if the manifold (and the Riemannian metric) is smooth. Therefore, for any (continuous) curve γ , the one-parameter family of Killing fields $L_t = H_{\gamma_t, \gamma'_t}$ satisfies conditions (4.1) and (4.3), namely, by point (i), the existence of a local isometric evolution for any observer.

(iii) Immediately follows by Proposition 4.2 and point (ii). \square

4.2. Dethermalization for conformal fields. Besides the geometric question of existence of the curve $t \mapsto \lambda_t \in \text{Conf}(\mathcal{M})$, there is a second existence problem if we want to describe the local dynamics in quantum field theory. Indeed, it is not obvious that the local maps λ_t are unitarily implemented, or give rise to automorphisms of the net. This is clearly the case of a conformally covariant theory, but not the general case.

The previous discussion on local evolutions shows that the evolution may change if we replace the original metric with another metric in the same conformal class. We shall show that, with a suitable choice of the new metric, the original observer will become an inertial observer in a (locally) flat spacetime. Therefore, in a conformal quantum field theory, the local evolution will be implemented by a one-parameter group with positive generator w.r.t. which the vacuum state is a ground state.

We mention the analysis contained in [17], where the authors classify the global conformal vacua on a conformally flat spacetime in terms of the global timelike Killing vector fields. In the Minkowski spacetime there is only one global timelike Killing vector field, while for other “small” spacetimes one may have two nonequivalent Killing vector fields, as is the case of the Rindler wedge subregion where the boost flow is also timelike. The two Killing vector fields give rise to different vacua, and the vacuum for the Minkowskian Killing vector field is thermalized w.r.t. the second Killing evolution. Our construction represents a converse to this procedure: starting with dS^d , where the global Killing vector field is unique and the de Sitter vacuum is thermalized, we restrict to a smaller spacetime where a global dethermalizing conformal Killing flow exists.

From a classical point of view then, the dethermalization is realized by replacing the original dynamics with a new ‘conformal’ dynamics.

Let us note that such a change of the dynamics implies in particular a change in the time parametrization. Of course the absence of a preferred proper time parametrization occurs if the conformal structure alone is considered.

As we shall see in the next sections, the evolutions λ will give rise only to a quasi-covariant dynamics in the sense of [15] for general (non conformal) quantum fields.

As seen in Subsection 3.1, there exists a conformal diffeomorphism Ψ between the steady-state universe subspace \mathcal{N} of dS^d containing a given complete causal geodesics γ and the semispace $M_+^d = \{(\mathbf{x}, t) \in M^d, t > 0\}$ in the Minkowski space, mapping γ to a causal geodesics $\tilde{\gamma}$. However $\tilde{\gamma}$ is not complete, and can be identified with the half-line $\{\mathbf{x} = 0, t > 0\}$ in the timelike case, and with the half-line $\{x_1 = t, x_i = 0, i > 1, t > 0\}$ in the lightlike case. Therefore we get the following.

Proposition 4.5. *If we replace the metric on \mathcal{N} with the pull back via Ψ of the flat metric on M_+^d , there exists a local evolution μ_t , $t > 0$, from \mathcal{N} into itself, given by the pull back of the time translations.*

Theorem 4.6. *Let \mathcal{A} be a conformal net on dS^d and W a wedge causally generated by a geodesic observer γ . Then:*

- (a) *The local isometric evolution λ corresponding to the de Sitter metric is indeed global, there exists one-parameter unitary group U on the Hilbert*

space implementing λ and the vacuum is a thermal state at the Gibbons-Hawking temperature w.r.t. U .

- (b) The local isometric evolution μ corresponding to the flat metric is unitarily implemented, namely there exists a one-parameter unitary group V on the Hilbert space such $V(t)$ implements μ_t for $t > 0$, and the vacuum is a ground state w.r.t. V . If we extend the net \mathcal{A} to a conformal net $\tilde{\mathcal{A}}$ on the static Einstein universe, then $V(t)$ acts covariantly on $\tilde{\mathcal{A}}$ for every $t \in \mathbb{R}$.

Proof. The first statement is simply a reformulation of assumption **c**) in Section 2. Concerning the second statement, note that μ extends to a global flow on E^d which is implemented by a one parameter group V with positive generator. The thesis is then immediate. \square

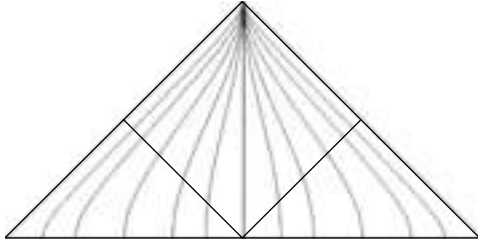


Fig. 2. The flow lines of the isometric evolution λ in the wedge contained in the steady-state universe.

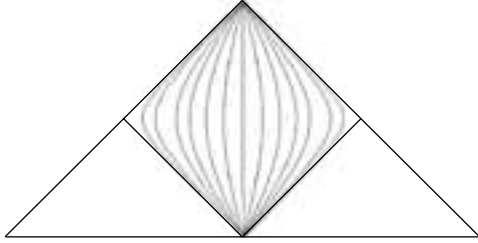


Fig. 3. The flow lines of the dethermalizing evolution μ in the steady-state universe.

Remark 4.7. In the conformal case, the dethermalizing evolution is not unique. In fact we may identify dS^d with a rectangle in the Einstein universe (cf. eq. (3.11)), and then consider the corresponding metric on it. Again, the new evolution, which is given by time translations in E^d , is dethermalized.

4.3. Dethermalization with noncommutative flows. As anticipated, we construct here a quasi-covariant dynamics corresponding to the geometric dynamics described above, showing that the vacuum vector becomes a ground state w.r.t. this dynamics.

Our flow will be noncommutative in the sense it give a noncommutative dynamical system, indeed it is a flow on a quantum algebra of observables, although it will retain a partial geometric action.

We begin with a no-go result.

Proposition 4.8. *Let U be a non-trivial unitary representation of $SO_0(d, 1)$, $d \geq 2$, and u the associated infinitesimal representation of the Lie algebra $so(d, 1)$. The following are equivalent:*

- (i) *There exists a non-zero $L \in so(d, 1)$ such that $u(L)$ is a positive or negative operator;*
- (ii) *$d = 2$ and U is the direct sum of irreducible representations that are either the identity or belong to the discrete series of $SO_0(2, 1)$ ($\simeq PSL(2, \mathbb{R})$). In this case L belongs to the cone generated by the translation generators.*

Proof. The set \mathfrak{P} of $L \in so(d, 1)$ such that $u(L)$ is a positive operator is a convex cone of $so(d, 1)$, which is globally stable under the adjoint action of $SO_0(d, 1)$, and $\mathfrak{P} \cap -\mathfrak{P} = 0$ because $so(d, 1)$ is a simple Lie algebra.

Now every element $L \in so(d, 1)$ can be written as a sum $L = R + K$, where $R \in so(d)$ and K is the generator of a boost one-parameter subgroup.

Let then L belong to \mathfrak{P} and assume $d > 2$. We can then choose a rotation $r \in SO(d)$ such that $\text{Ad } r(K) = -K$. Set $R' \equiv \text{Ad } r(R) \in so(d)$. Since $L' \equiv \text{Ad } r(L) \in \mathfrak{P}$, the element

$$L + L' = R + K + \text{Ad } r(R) + \text{Ad } r(K) = R + R'$$

belongs to \mathfrak{P} and to $so(d)$, so it is enough to show that $\mathfrak{P} \cap so(d) = \{0\}$. Indeed if $R'' \in so(d)$, $d > 2$, we can choose a rotation r such that $\text{Ad } r(R'') = -R''$, thus $R'' = 0$ if $R'' \in \mathfrak{P}$.

We thus conclude that $d = 2$. Now every non-zero $L \in SO_0(2, 1) \simeq PSL(2, \mathbb{R})$ is (conjugate to) the generator of either a boost, or translation, or rotation one-parameter group. If L is a boost generator, then L is conjugate to $-L$ as above, thus $L \notin \mathfrak{P}$. The positivity of $u(L)$, L a translation generator, is equivalent to the positivity of $u(L)$, L a rotation generator (see e.g. [37]) and is equivalent to U to be a direct sum of representations in the discrete series of U and, possibly, to the identity [42]. \square

The next corollary states that the existence of a dethermalized covariant one-parameter dynamics is possible only if $d = 2$ and implies conformal covariance.

Corollary 4.9. *Given a local net \mathcal{A} on the de Sitter space, assume there is a one parameter group in $SO_0(d, 1)$ which has positive generator in the covariance representation. Then \mathcal{A} is conformally covariant.*

Proof. Assume the net is not conformally covariant. By the Proposition above, this implies $d = 2$. Then, Corollary 5.16 shows that positive energy representations in the two-dimensional case imply conformal covariance. \square

Now we turn to a geodesic observer γ , and denote by W the wedge generated by the complete geodesic, by \mathcal{N} the steady-state universe containing W , by λ the Killing flow corresponding to the geodesic γ , by μ_t , $t > 0$, the conformal evolution of \mathcal{N} described above. Let us observe that the time is reparametrized, namely $\tilde{\gamma}_t = \gamma_{\log t} = \mu_{t-1}\gamma_0$. We also denote by R the spacetime reflection mapping W to its spacelike complement W' .

Theorem 4.10. *Let \mathcal{A} be a net of local algebras on the de Sitter spacetime. Then there exists a unique one-parameter unitary group V with the following properties:*

- (i) Ω is a ground state w.r.t. V ;
- (ii) V implements a quasi covariant dynamics for the regions $\mu_t(W)$, $t \geq 0$, namely $V(t)\mathcal{A}(W)V(-t) = \mathcal{A}(\mu_t W)$, $t \geq 0$;
- (iii) *Partial localization for negative times:* $V(-t)\mathcal{A}(W)V(t) = \mathcal{A}(R\mu_t W)'$, $t \geq 0$.

Proof. By the geodesic KMS property, we get that $\mathcal{A}(\mu_1(W)) \subset \mathcal{A}(W)$ is a half-sided modular inclusion. Therefore the theorem of Wiesbrock [55] gives a one-parameter group V with positive generator such that $V(1)\mathcal{A}(W)V(-1) = \mathcal{A}(\mu_1(W))$ and satisfying the Borchers commutation relations

$$\begin{aligned} U(\lambda_t)V(s)U(\lambda_{-t}) &= V(e^t s) \\ U(R)V(s)U(R) &= V(-s). \end{aligned}$$

Then (i) is obvious, and the above relations give

$$\begin{aligned} V(t)\mathcal{A}(W)V(-t) &= U(\lambda_{\log t})V(1)U(\lambda_{-\log t})\mathcal{A}(W)U(\lambda_{\log t})V(-1)U(\lambda_{-\log t}) \\ &= \mathcal{A}(\mu_t W), t \geq 0, \end{aligned}$$

namely (ii). Property (iii) follows in an analogous way.

The uniqueness now follows by the uniqueness for one-parameter groups with Borchers property [4] in the following lemma. \square

Remark 4.11. By the above theorem we have the following localization properties for the noncommutative flow $\text{Ad } V$:

(i) If \mathcal{L} is a region contained in the steady state universe subregion \mathcal{N} of dS and $\mathcal{L} = \mu_s W$ for some $s \geq 0$, then $\mu_t \mathcal{L} \subset dS^d$ if and only if $t \in [-s, +\infty)$ and, for such t , $\text{Ad } V(t)\mathcal{A}(\mathcal{L}) = \mathcal{A}(\mu_t \mathcal{L})$. Analogously, considering μ_t as a global transformation on E^d acting on dS^d by restriction, using the geometric action of J , and Borchers commutation relations $JV(t)J = V(-t)$, if \mathcal{L} is a region in the complement of \mathcal{N} and $\mathcal{L} = \mu_{-s} W'$ for some $s \geq 0$, then $\mu_t \mathcal{L} \subset dS$ if and only if $t \in (-\infty, s]$ and, for such t , $\text{Ad } V(t)\mathcal{A}(\mathcal{L}) = \mathcal{A}(\mu_t \mathcal{L})$.

(ii) If \mathcal{O} is a double cone contained in \mathcal{N} , there exists $s > 0$ such that, for any $s' \geq s$, $\mathcal{O} \subset (\mu_{-s'} W')'$. Therefore, for any $t \in \mathbb{R}$,

$$(4.4) \quad \text{Ad } V(t)\mathcal{A}(\mathcal{O}) \subset \begin{cases} \mathcal{A}(\mu_{t-s} W')' & \text{if } t - s \leq 0 \\ \mathcal{A}(\mu_{t-s} W) & \text{if } t - s \geq 0. \end{cases}$$

Assuming Haag duality on dS^d we then get

$$(4.5) \quad \text{Ad } V(t)\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mu_{t-s} W \cap dS^d)$$

for any double cone $\mathcal{O} \subset \mathcal{N}$; note that $\mu_{t-s} W \cap dS^d$ has non empty spacelike complement in dS^d . Analogous localization properties hold if \mathcal{O} is contained in $dS^d \setminus \mathcal{N}$.

Localization results for any double cone $\mathcal{O} \subset dS^d$ would then follow by a form of strong additivity.

Let us remark that more stringent localization properties would indeed imply a complete geometrical action [39].

Lemma 4.12. *Let \mathcal{P} be a von Neumann algebra on a Hilbert space \mathcal{H} with cyclic and separating vector $\Omega \in \mathcal{H}$. Let V_1 and V_2 be Ω -fixing one-parameter unitary groups on \mathcal{H} such that $V_k(t)\mathcal{P}V_k(-t) \subset \mathcal{P}$, $t \geq 0$, ($k = 1, 2$) and $V_1(1)\mathcal{P}V_1(-1) = V_2(1)\mathcal{P}V_2(-1)$. Suppose that the generators of V_1 and V_2 are positive. Then $V_1 = V_2$.*

Proof. By Borchers theorem [4] we have $\Delta^{is}V_k(t)\Delta^{-is} = V_k(e^{-2\pi s t})$, $t, s \in \mathbb{R}$, where Δ is the modular operator associated with (\mathcal{P}, Ω) . We then have $\text{Ad } V_1(t)(\mathcal{P}) = \text{Ad } V_2(t)(\mathcal{P})$, $t \geq 0$, because

$$\text{Ad } V_k(e^{-2\pi s})(\mathcal{P}) = \text{Ad } \Delta^{is}V_k(1)\Delta^{-is}(\mathcal{P}) = \text{Ad } \Delta^{is}V_k(1)(\mathcal{P}) = \text{Ad } \Delta^{is}(\mathcal{P}_1), \quad s \in \mathbb{R}.$$

Then $Z(t) \equiv V_2(-t)V_1(t)$, $t \geq 0$, is Ω -fixing and implements an automorphism of \mathcal{P} , thus commutes with Δ^{is} . On the other hand $\Delta^{is}Z(t)\Delta^{-is} = Z(e^{-2\pi s t})$, due to the above commutation relations, so $Z(t) = Z(e^{-2\pi s t})$ for all $t \geq 0$ and all $s \in \mathbb{R}$. Letting $s \rightarrow \infty$ we conclude that $Z(t) = 1$, that is $V_1(t) = V_2(t)$, for $t \geq 0$ and thus for all $t \in \mathbb{R}$ because $V_k(-t) = V_k(t)^*$. \square

The following table summarizes the basic structure in the above discussion.

| <i>space</i> | <i>orbit</i> | <i>flow</i> | ω | <i>orbit</i> | <i>flow</i> | ω |
|--------------|--------------|--------------|----------|--------------|-------------|----------|
| Minkowski | geodesic | translations | ground | hyperbola | boosts | KMS |
| de Sitter | geodesic | boosts | KMS | geodesic | μ | ground |

5. TWO-DIMENSIONAL DE SITTER SPACETIME

5.1. Geometric preliminaries. Let us assume that dS^2 is oriented and time-oriented. Following Borchers [5], a wedge W at the origin (namely a wedge whose edge contains the origin) in the Minkowski space M^d is determined by an ordered pair of linearly independent future-pointing lightlike vectors ℓ_1, ℓ_2 ; W is the open cone spanned by $\ell_1, -\ell_2$ and vectors orthogonal to ℓ_1, ℓ_2 (in the Minkowski metric). In order to make this correspondence 1 : 1 one can normalize the vectors in such a way that their time-component is 1. Clearly such a pair determines and is determined by the $(d - 2)$ oriented hyperplane which is orthogonal to ℓ_1 and ℓ_2 w.r.t. the Minkowski metric (the edge of the wedge). In particular, when $d = 3$, it is determined by an oriented line ζ through the origin, e.g. by requiring that $\ell_1, -\ell_2, v$ determine the orientation in M^3 when v is an oriented vector in ζ . Denoting by x, \tilde{x} the intersection points of ζ with dS^2 , with x preceding \tilde{x} according to the orientation, it is clear that \tilde{x} is the symmetric of x w.r.t the origin, therefore x determines the wedge, so the map $x \mapsto W(x)$ is a bijection between points of dS^2 and wedges.

Any point x in dS^2 determines two lightlike lines given by the intersection of dS^2 with the tangent plane at x (the ruled lines through x of the hyperboloid). Let us denote them by $h_r(x), h_l(x)$ in such a way that, if v_r, v_l are future pointing vectors in $h_r(x), h_l(x)$ respectively, the pair (v_r, v_l) determines the given orientation of dS^2 .

Now let us consider an observer generating the wedge $W(x)$. Then, the sets $h_r(x) = h_r(x) \cup h_r(\tilde{x}), h_l(x) = h_l(x) \cup h_l(\tilde{x})$ form a bifurcated Killing horizon for dS^2 [35], see also [27]), the Killing flow being the one-parameter group of pure Lorentz transformations associated to the wedge $W(x)$. and the set $\mathfrak{H} = h_r(x) \cup h_l(\tilde{x})$ is the event horizon for $W(x)$, which splits in the two components $\mathfrak{H}_+ = h_r(x)$ $\mathfrak{H}_- = h_l(\tilde{x})$.

Clearly any point $x \in dS^2$ determines a partition of the space into 6 disjoint regions: $W(x)$ (the right of x), $W(\tilde{x})$ (the left of x), $V_+(x)$ (the closed future cone at x), $V_-(x)$ (the closed past cone at x), $V_+(\tilde{x})$ (the closed future cone at \tilde{x}), $V_-(\tilde{x})$ (the closed past cone at \tilde{x}).

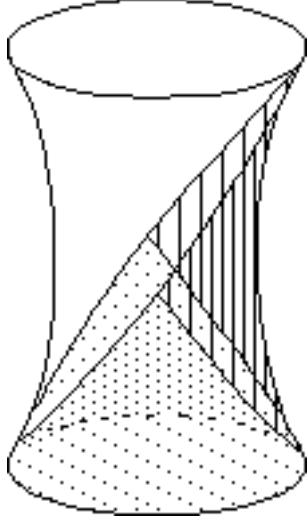


Fig. 4. Two-dimensional de Sitter space. The whole marked area is the steady-state universe, whose boundary is the event horizon. The striped area is the wedge region (static de Sitter space-time), whose boundary is the black-hole horizon.

Lemma 5.1. *Two wedges $W(x)$, $W(y)$ have non-empty intersection if and only if y belongs to $W(x) \cup W(\tilde{x}) \cup V_+(x) \cup V_-(x)$.*

Proof. The “if” part is obvious. Concerning the “only if” part, assume that y is in the future of \tilde{x} . Then $W(y)$ is contained in the future of $W(\tilde{x})$. Since the latter is the region in the future of the $h_A(x)$ horizon, while $W(x)$ is contained in the past of $h_A(x)$, the thesis follows. \square

Now we may characterize the sets that are intersections of wedges.

Lemma 5.2. *In dS^2 , every non-empty open region \mathcal{O} given by an intersection of wedges, is indeed an intersection of two (canonically determined) wedges, or, equivalently, $\mathcal{O} \in \tilde{\mathcal{K}}$.*

Proof. Let \mathcal{O} be an open region given by intersection of wedges, and let $X(\mathcal{O})$ the set of points x such that $W(x) \supset \mathcal{O}$. Endow $X(\mathcal{O})$ with the partial order relation of being “to the right”, namely $x > y$ if $x \in W(y)$. If $x, y \in X(\mathcal{O})$ are not comparable, since $\mathcal{O} \subset W(x) \cap W(y)$, Lemma 5.1 implies that one is in the future of the other. If x is in the future of y , define $x \vee y$ as the intersection of $h_l(x)$ with $h_r(y)$. Clearly, if $\mathcal{O} \subset W(x) \cap W(y)$, then $\mathcal{O} \subset W(x \vee y)$. Therefore $X(\mathcal{O})$ is directed. Since \mathcal{O} is open, the supremum of any ordered subset in $X(\mathcal{O})$ belongs to $X(\mathcal{O})$, hence there exists a maximal element. Directedness implies that such maximal element is indeed a maximum $L(\mathcal{O})$ (the leftmost point of the closure of \mathcal{O}). Analogously we get a minimum $R(\mathcal{O})$ among the points y such that $W(\tilde{y}) \supset \mathcal{O}$ (the rightmost point of the closure of \mathcal{O}). Clearly $\mathcal{O} = W(L(\mathcal{O})) \cap W(\tilde{R}(\mathcal{O}))$. Such a set is the double cone (possibly degenerate, i.e. $\mathcal{O} \in \tilde{\mathcal{K}}$) generated by the points $F(\mathcal{O}) = h_r(L(\mathcal{O})) \cap h_l(R(\mathcal{O}))$, $P(\mathcal{O}) = h_l(L(\mathcal{O})) \cap h_r(R(\mathcal{O}))$. \square

We shall call $L(\mathcal{O}), R(\mathcal{O})$ the spacelike endpoints of \mathcal{O} , and $P(\mathcal{O}), F(\mathcal{O})$ the timelike endpoints of \mathcal{O} .

5.2. Geometric holography. Now we fix the event horizon as the intersection of the plane $x_0 = y_0$ with the de Sitter hyperboloid, the two components being $\mathfrak{H}_\pm = \{(t, t, \pm\rho) : t \in \mathbb{R}\}$. In the two-dimensional case, the orientation preserving isometry group of the de Sitter spacetime is isomorphic to $SO_0(2, 1)$. On the other hand $SO_0(2, 1)$ is isomorphic to $PSL(2, \mathbb{R})$ and acts on (the one-point compactification of) \mathfrak{H}_+ or \mathfrak{H}_- . We shall construct holographies based on this equality.

The Möbius group is the semidirect product of $PSL(2, \mathbb{R})$ with \mathbb{Z}_2 . Let us chose the following generators for its Lie algebra $sl(2, \mathbb{R})$:

$$(5.1) \quad D = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The following commutation relations hold:

$$(5.2) \quad [D, T] = T, \quad [D, A] = -A, \quad [T, A] = D.$$

We consider also the following orientation reversing element of the Möbius group:

$$(5.3) \quad r = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let us observe that the following relations hold:

$$(5.4) \quad rDr = D, \quad rTr = -T, \quad rAr = -A.$$

We shall denote by β the usual action of the Möbius group on $\mathbb{R} \cup \{\infty\}$ as fractional linear transformations. Then $\beta(r)$ implements the reflection $x \mapsto -x$, $\beta(\exp(tD))$ implements the dilations, $\beta(\exp(tT))$ implements the translations, and $\beta(\exp(tA))$ implements the anti-translations (see [26]).

Now we consider the two immersions

$$\begin{aligned} \psi_{\pm} : \mathbb{R} &\rightarrow \mathfrak{H}_{\pm} \subset dS^2 \\ t &\mapsto (t, t, \pm\rho) \end{aligned}$$

of the real line in dS^2 as \pm -horizon, and will look for actions α_{\pm} of the Möbius group on dS^2 with the following property: whenever $\alpha_{\pm}(g)$ preserves \mathfrak{H}_{\pm} , then

$$(5.5) \quad \alpha_{\pm}(g)\psi_{\pm}(t) = \psi_{\pm}(\beta(g)t).$$

Lemma 5.3. *The previous requirement determines α_{\pm} uniquely, in particular we have*

$$\begin{aligned} \alpha_+(D) &= \alpha_-(D) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \alpha_+(T) &= -\alpha_-(T) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \\ \alpha_+(A) &= -\alpha_-(A) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \\ \alpha_+(r) &= \alpha_-(r) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

where α_{\pm} also denote the associated actions of $sl(2, \mathbb{R})$. Moreover, the following relation holds:

$$(5.6) \quad \alpha_-(g) = \alpha_+(rgr).$$

Proof. It is easy to see that the subgroup (globally) stabilizing \mathfrak{H}_+ coincides with the subgroup stabilizing \mathfrak{H}_- and is generated by $\alpha_+(\exp(tD))$, $\alpha_+(\exp(tT))$, and $\alpha_+(r)$, as they are defined in the statement, therefore the identification is forced by eq. (5.5) for these elements. Eq. (5.2) implies then the formula for $\alpha_+(A)$. The

proof for α_- is analogous. Relation (5.6) immediately follows from the previous equations and relations (5.4). \square

Remark 5.4. By Lemma 5.3, it follows that $\alpha_+(T)$ and $\alpha_-(T)$ have opposite signs, thus, given a unitary representation U of $SO_0(2, 1)$, the generator of $U(\alpha_+(\exp(tT)))$ is positive if and only if the generator of $U(\alpha_-(\exp(tT)))$ is negative.

Now we define two maps Φ_\pm from the set \mathcal{W} of wedges in dS^2 to the set \mathcal{I} of open intervals in (the one-point compactification of) \mathbb{R} such that, for any element g in the Möbius group and any wedge W , one has

$$(5.7) \quad \Phi_\pm(\alpha_\pm(g)W) = \beta(g)\Phi_\pm(W).$$

Proposition 5.5. *Let $W \in \mathcal{W}$. Then $\partial W \cap \mathfrak{H}_+ \neq \emptyset \Leftrightarrow \partial W \cap \mathfrak{H}_- \neq \emptyset$. The maps Φ_\pm are uniquely determined by the further requirement that, for any such wedge,*

$$(5.8) \quad \Phi_\pm(W) = \psi_\pm^{-1}(\partial W).$$

Moreover they satisfy

$$(5.9) \quad \Phi_\pm(W') = \Phi_\pm(W)'$$

$$(5.10) \quad \Phi_+(W) = \beta(r)\Phi_-(W'),$$

where the prime ' denotes the spacelike complement in dS^2 and the interior of the complement in S^1 .

Proof. Let us construct Φ_+ , the construction of Φ_- being analogous. For notational simplicity we shall drop the subscript $+$ in the rest of the proof. Let W_0 be the wedge $W(0, 1, 0)$, according to the previous description. Since the Lorentz group acts transitively on wedges, property (5.7) may be equivalently asked for W_0 only. Now eq. (5.8) implies $\Phi(W_0) = I_0$, where I_0 denotes the positive half line, hence we only have to test that equation $\Phi(\alpha(g)W_0) = \beta(g)I_0$ makes Φ well defined. This is equivalent to show that if $\alpha(g)W_0 = W_0$, then $\beta(g)I_0 = I_0$. The stabilizer of W_0 is easily seen to be generated by $\alpha(\exp(tD))$ and $\alpha(\hat{r})$, where $\hat{r} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, since

$$\alpha(\hat{r}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

A direct computation shows that $\beta(\exp(tD))$ and $\beta(\hat{r})$ stabilize I_0 . Now we show that eq. (5.8) is always satisfied. Indeed, let $\partial W \cap \mathfrak{H} \neq \emptyset$, namely either $W = W(x)$ or $W = W(\tilde{x})$, with $x \in \mathfrak{H}$. Then there exists g stabilizing \mathfrak{H} , either of the form $\exp(sT)$, or of the form $\exp(sT)r$, such that $W = \alpha(g)W_0$. Eq. (5.5) then implies the thesis.

Now we prove the (5.9). Indeed, by (5.7), it is enough to prove it for only one wedge, e.g. W_0 , where it follows immediately by (5.8). Concerning (5.10), we have

$$\begin{aligned} \Phi_+(\alpha_+(g)W_0) &= \beta(g)\Phi_+(W_0) = \beta(g)\Phi_-(W_0) \\ &= \Phi_-(\alpha_-(g)W_0) = \Phi_-(\alpha_-(r)\alpha_+(g)\alpha_-(r)W_0) = \beta(r)\Phi_-(\alpha_+(g)W'_0). \end{aligned}$$

\square

Let us observe that the above mentioned map trivially preserves inclusions, indeed no wedge is properly contained in another wedge of dS^d , while the inverse map does not.

Now we may pass to points. Indeed any point in dS^2 corresponds to a wedge: $x \mapsto W(x)$. Also, any interval in the one-point compactification of \mathbb{R} determines its leftmost extreme: $I \mapsto \ell(I)$. Then we may define point maps as follows:

$$(5.11) \quad \varphi_{\pm}(x) = \ell(\Phi_{\pm}(W(x))).$$

Lemma 5.6. *The point maps φ_{\pm} are equivariant, namely*

$$(5.12) \quad \varphi_{\pm}(\alpha_{\pm}(g)x) = \beta(g)\varphi_{\pm}(x).$$

Proof. Assume g to be orientation preserving. Then $\alpha_{\pm}(g)W(x) = W(\alpha_{\pm}(g)x)$ and $\ell(\beta(I)) = \beta(\ell(I))$. Therefore the result follows from (5.7). Assume now g to be orientation reversing. Given $x \in dS^2$, we may write g as $h_1 r h_2$, where $\alpha_+(h_2)x = x_0 \equiv (0, 0, \rho)$. Then eq. (5.12) reduces to $\varphi_+(\alpha_+(r)x_0) = \beta(r)\varphi_+(x_0)$, which is obvious. The proof in the $-$ case is analogous. \square

Theorem 5.7. *The wedge maps Φ_{\pm} are induced by the point maps φ_{\pm} , namely $\Phi_{\pm}(W) = \{\varphi_{\pm}(x) : x \in W\}$. The point maps φ_{\pm} are given by the holographic projections*

$$x \in dS^2 \mapsto h_{\mp}(x) \cap \mathfrak{H}_{\pm},$$

where \mathfrak{H}_{\pm} are identified with \mathbb{R} as before.

Proof. We prove the second statement first. Indeed, it is sufficient to show that the preimage under φ_{\pm} of a point t in \mathbb{R} is the ruled line $h_{\mp}(\psi_{\pm}(t))$. Eq. (5.12) implies that this is simply the α_{\pm} -orbit of the β -stabilizer of t , and that we may check the property for one point only, say $t = 0$. The elements of $SO(2, 1)$ β -stabilizing 0 but not α_{\pm} -stabilizing $\psi_{\pm}(0)$ are of the form $\exp(sT)$, and the orbit of $\alpha_{\pm}(\exp(sT))$ at $\psi_{\pm}(0)$ is exactly $h_{\mp}(\psi_{\pm}(0))$.

Now we prove the first statement in the $+$ case. Let $x \in W$. By equivariance, we can move x and W in such a way that $W = W(\psi_+ t)$, and $x \in h_-(\psi_+(0))$. Then the statement becomes $\varphi_+(x) \in \Phi_+(W)$, i.e. $t < \varphi_+(x)$, but this is obvious since $x \in W$. The proof for the $-$ case is analogous. \square

The maps φ_{\pm} may be considered as geometric holographies, namely projection maps from the de Sitter space to (some part of) the horizon preserving the causal structure and intertwining the symmetry group actions. Of course one can construct holography maps onto the conformal boundary as well, simply associating with any x the intersection of $h_{\pm}(x)$ with the conformal boundary.

5.3. Pseudonets. By a *local conformal pseudonet* \mathcal{B} on a Hilbert space \mathcal{H} (or simply a local pseudonet) we shall mean here a map \mathcal{B} from the (proper, open, non-empty) intervals \mathcal{I} of S^1 to von Neumann algebras on \mathcal{H} with the following properties:

- *Möbius covariance.* There exists a unitary representation U of $PSL(2, \mathbb{R})$ on \mathcal{H} such that $U(g)\mathcal{B}(I)U(g)^{-1} = \mathcal{B}(gI)$, $g \in PSL(2, \mathbb{R})$, $I \in \mathcal{I}$.
- *Vacuum with Reeh-Schlieder property.* There exists a unit, U -invariant vector Ω , cyclic for each $\mathcal{B}(I)$.
- *Interval KMS property.* $\Delta_I^{it} = U(\Lambda_I(-2\pi t))$, $I \in \mathcal{I}$, where Δ_I is the modular operator associated with $(\mathcal{B}(I), \Omega)$ and Λ_I is the one-parameter subgroup of $PSL(2, \mathbb{R})$ of special conformal transformations associated with I , see [10].
- *Locality.* $\mathcal{B}(I)$ and $\mathcal{B}(I')$ commute elementwise for every $I \in \mathcal{I}$ (with I' the interior of $S^1 \setminus I$).

Note that we do *not* assume *positivity of the energy* (or negativity of the energy) nor *isotony* (or anti-isotony).

Given a local pseudonet \mathcal{B} on the Hilbert space \mathcal{H} , let J be the canonical anti-unitary from \mathcal{H} to conjugate Hilbert space $\overline{\mathcal{H}}$. We define the *conjugate pseudonet* $\overline{\mathcal{B}}$ on $\overline{\mathcal{H}}$ by

$$\overline{\mathcal{B}}(I) = J\mathcal{B}(I')J, \quad \overline{U}(g) = JU(g)J, \quad \overline{\Omega} = J\Omega.$$

We may define $\overline{\mathcal{B}}$ directly on \mathcal{H} with the same vacuum vector by choosing a reflection r on S^1 associated with any given interval I_0 (say $r : z \mapsto -z$) and putting

$$\overline{\mathcal{B}}(I) = \mathcal{B}(rI')$$

with the covariance unitary representation \overline{U} given by

$$\overline{U}(g) = U(rgr), \quad g \in PSL(2, \mathbb{R}).$$

In this case $\overline{\mathcal{B}}$ depends on the choice of r , but is well defined up to unitary equivalence. The second conjugate of \mathcal{B} is equivalent to \mathcal{B} . \mathcal{B} is isotonic iff $\overline{\mathcal{B}}$ is anti-isotonic, and \mathcal{B} has positive energy iff $\overline{\mathcal{B}}$ has negative energy. Note that $\overline{\mathcal{B}}$ is defined also if Ω is not cyclic.

Theorem 5.8. *Let \mathcal{B} be a local pseudonet.*

- (i) *Haag duality holds: $\mathcal{B}(I)' = \mathcal{B}(I')$, $I \in \mathcal{I}$.*
- (ii) *If Ω is unique U -invariant, then each $\mathcal{B}(I)$ is a type III_1 factor.*
- (iii) *\mathcal{B} is isotonic (resp. anti-isotonic) iff it has positive energy (resp. negative energy).*

Proof. (i) By locality, $\mathcal{B}(I')$ is a von Neumann subalgebra of $\mathcal{B}(I)'$, globally invariant with respect to the modular group $\text{Ad}\Delta_I^{-it}$ of $\mathcal{B}(I)'$, hence $\mathcal{B}(I') = \mathcal{B}(I)'$ by Takesaki theorem due to the Reeh-Schlieder property of Ω .

(ii) If Ω is unique U -invariant, then, as in [26], $\text{Ad}\Delta_I^{it}$ is ergodic on $\mathcal{B}(I)$, and this entails the III_1 -factor property.

(iii) If \mathcal{B} is isotonic, then positivity of the energy follows from the interval KMS property, see e.g. [27]. Conversely, if U has positive energy, let us prove isotony. Clearly it is enough to prove isotony for pairs $\tilde{I} \subset I$ having one extreme point in common, and, by $SO(2, 1)$ covariance, we need only one pair, say $I = (0, \infty)$, $\tilde{I} = (1, \infty)$, namely it is enough to show that translations $T(t)$ implement endomorphisms of $\mathcal{B}(I)$ for positive t . By a classical argument, positivity is equivalent to the positivity of the self-adjoint generator of the translations $T(t)$, therefore we have the four ingredients of the Borchers theorem: a vector Ω , the vacuum, which is invariant for the representation, hence for the modular group Δ_I^{it} of $\mathcal{B}(I)$ and for $U(T(t))$, the commutation relations between these one-parameter groups, the positivity of the generator of translations, and an (expected) implementation of $\mathcal{B}(I)$ -endomorphisms by $U(T(t))$ for positive t . Davidson [19] proved that the last property follows from the first three ones if the following holds: there exists an $\varepsilon > 0$ such that the vacuum is cyclic for the set $\mathcal{B}(\varepsilon)$ consisting of all the $x \in \mathcal{B}(I)$ such that, for all $t \in (0, \varepsilon)$, $U(T(t))xU(T(-t))$ is in $\mathcal{B}(I)$ (Theorem 3 *ibid.*). Now $\mathcal{B}(\tilde{I}) \subset \mathcal{B}(1)$, hence the cyclicity follows.

The equivalence between anti-isotony and negative energy is obtained by considering the conjugate pseudonet. \square

Let us define the “isotonized” nets associated with \mathcal{B} , resp. $\overline{\mathcal{B}}$:

$$\mathcal{B}_+(I_0) = \bigcap_{I \supset I_0} \mathcal{B}(I), \quad \mathcal{B}_-(I_0) = \bigcap_{I \supset I_0} \overline{\mathcal{B}}(I).$$

Then \mathcal{B}_\pm is isotonic, thus it has positive energy (on the vacuum cyclic subspace). Moreover $\mathcal{B}_+(I)$ is globally invariant w.r.t. $\text{Ad}\Delta_I^{it}$ thus, by Takesaki theorem, there

is a vacuum preserving normal conditional expectation from $\mathcal{B}(I)$ onto $\mathcal{B}_+(I)$. It is easy to check that

$$\overline{\mathcal{B}}_-(I_0) = \bigcap_{I \subset I_0} \mathcal{B}(I),$$

hence $\overline{\mathcal{B}}_-$ is expected in \mathcal{B} and $\mathcal{B}_+(I) \vee \overline{\mathcal{B}}_-(I) \subset \mathcal{B}(I)$.

Proposition 5.9. *If Ω is unique U -invariant, we have the von Neumann tensor product splitting*

$$\mathcal{B}_+(I_1) \vee \mathcal{B}_-(I_2) = \mathcal{B}_+(I_1) \otimes \mathcal{B}_-(I_2).$$

Proof. First we show that $\mathcal{B}_+(I_1)$ and $\mathcal{B}_-(I_2)$ commute for any $I_1, I_2 \in \mathcal{I}$. As \mathcal{B}_+ is a net, it is additive and we may assume that $rI_1 \cup I_2$ has non-empty complement. We may then enlarge I_2 in such a way that $rI_1 \subset I_2$. Then $\mathcal{B}_+(I_1) \subset \mathcal{B}(I_1)$, and $\mathcal{B}_-(I_2) \subset \overline{\mathcal{B}}(I_2) = \mathcal{B}(rI_2)' \subset \mathcal{B}(I_1)'$, namely they commute. Then, as in Theorem 5.8 (ii), the von Neumann algebras $\mathcal{B}_\pm(I)$ are factors, hence they generate a von Neumann tensor product by Takesaki's theorem [52]. \square

5.4. Holography and chirality. Let \mathcal{B} a local pseudonet on S^1 . Then we may associate with it a local net \mathcal{A} on the wedges of dS^2 as follows:

$$(5.13) \quad \mathcal{A}(W) \equiv \mathcal{B}(\Phi_+(W)).$$

Clearly, given a pseudonet on S^1 , we may obtain a net on the double cones of dS^2 by intersection, and such net will satisfy properties **a**), **b**), **c**) on a suitable cyclic subspace. Conversely, given a net on dS^2 , equation (5.13) gives rise to a pseudo net on S^1 .

Let us consider the following property:

- *Intersection cyclicity.* For any pair of intervals $I_1 \subset I_2$, the vacuum vector is cyclic for the algebra

$$(5.14) \quad \mathcal{B}(I_1, I_2) \equiv \bigcap_{I_1 \subset I \subset I_2} \mathcal{B}(I)$$

Theorem 5.10. *The map (5.13) gives rise to a natural bijective correspondence between:*

- *Haag dual nets \mathcal{A} on dS^2 (satisfying properties **a**), **b**), **c**), **d**) in Sect. 2.2)*
- *Local pseudonets \mathcal{B} on S^1 satisfying intersection cyclicity.*

Proof. We only have to check that, setting

$$\mathcal{A}(\mathcal{O}) = \bigcap_{W \supset \mathcal{O}} \mathcal{A}(W)$$

for any double cone \mathcal{O} , the intersection cyclicity is equivalent to the Reeh-Schlieder property for double cones. We shall show that

$$(5.15) \quad \mathcal{A}(\mathcal{O}) = \mathcal{B}(\varphi_+(\mathcal{O}), \beta(r)\varphi_-(\mathcal{O})').$$

Indeed, any double cone \mathcal{O} can be described as a Cartesian product: $\mathcal{O} = I_+ \times I_-$, where $I_\pm = \varphi_\pm(\mathcal{O})$. Therefore, $W \supset \mathcal{O}$ is equivalent to $\Phi_\pm(W) \supset I_\pm$. Setting $I = \Phi_+(W)$ and making use of (5.10), this is in turn equivalent to $I_+ \subseteq I \subseteq \beta(r)I'_-$. In particular, since any double cone is contained in some wedge, I_+, I_- give rise to a double cone $\mathcal{O} = I_+ \times I_-$ iff $I_+ \subseteq \beta(r)I'_-$. The thesis follows. \square

We showed that any Haag dual net on dS^2 can be holographically reconstructed from a pseudonet on S^1 . Now we address the question of when such a net is conformal. Assuming intersection cyclicity, let us denote by Δ_{I_1, I_2} the modular operator associated with $(\mathcal{B}(I_1, I_2), \Omega)$ for a pair of intervals $I_1 \subset I_2$.

Theorem 5.11. *Let \mathcal{B} be a local pseudonet on S^1 satisfying intersection cyclicity, \mathcal{A} the corresponding Haag dual net on dS^2 . Then \mathcal{A} is conformal if and only if, for any $I_1 \subset L_1 \subset L_2 \subset I_2$,*

$$(5.16) \quad \Delta_{I_1, I_2}^{it} \mathcal{B}(L_1, L_2) \Delta_{I_1, I_2}^{-it} = \mathcal{B}(\Lambda_{I_1}(-2\pi t)(L_1), \Lambda_{I_2}(2\pi t)(L_2))$$

Proof. Let us note that local modular covariance (for the inclusion $\tilde{\mathcal{O}} \subset \mathcal{O}$) can be rephrased, in view of equation 5.15, as

$$(5.17) \quad \Delta_{I_1, I_2}^{it} \mathcal{B}(L_1, L_2) \Delta_{I_1, I_2}^{-it} = \mathcal{B}(\varphi_+(\Lambda_{\mathcal{O}}(t)\tilde{\mathcal{O}}), \beta(r)\varphi_-(\Lambda_{\mathcal{O}}(t)\tilde{\mathcal{O}})'),$$

where \mathcal{O} and $\tilde{\mathcal{O}}$ are determined by $\varphi_+(\mathcal{O}) = I_1$, $\varphi_-(\mathcal{O}) = \beta(r)I_2'$, $\varphi_+(\tilde{\mathcal{O}}) = L_1$, $\varphi_-(\tilde{\mathcal{O}}) = \beta(r)L_2'$.

We want to show that for any double cone \mathcal{O} , and any $x \in \mathcal{O}$,

$$\varphi_+(\Lambda_{\mathcal{O}}(t)x) = \Lambda_I(t)\varphi_+(x)$$

where $I = \varphi_+(\mathcal{O})$. It is enough to show the property when $\partial\mathcal{O} \cap \mathfrak{H}_+$ is non empty, since any other double cone can be reached via a transformation in the de Sitter group. In this case, I is identified with $\partial\mathcal{O} \cap \mathfrak{H}_+$.

First we observe that

$$\varphi_+(\Lambda_{\mathcal{O}}(t)x) = \Lambda_{\mathcal{O}}(t)\varphi_+(x),$$

since, identifying de Sitter with Minkowski, $\Lambda_{\mathcal{O}}$ splits as the product of the action on the chiral components. Since both are Möbius transformations on \mathfrak{H}_+ leaving I globally invariant, they should coincide, possibly up a reparametrization. Finally, we find a conformal transformation leaving \mathfrak{H}_+ globally stable and mapping \mathcal{O} onto a wedge W , therefore it is enough to check the equality on a wedge, where it follows by equivariance (5.12).

As a consequence, whenever $\tilde{\mathcal{O}} \subset \mathcal{O}$, $\tilde{I} = \varphi_+(\tilde{\mathcal{O}})$, we get

$$\varphi_+(\Lambda_{\mathcal{O}}(t)\tilde{\mathcal{O}}) = \Lambda_I(t)\tilde{I}.$$

In an analogous way we get

$$\varphi_-(\Lambda_{\mathcal{O}}(t)\tilde{\mathcal{O}}) = \Lambda_{\varphi_-(\mathcal{O})}(t)\varphi_-(\tilde{\mathcal{O}}).$$

These equations show that relations (5.16) and (5.17) are equivalent, therefore the thesis follows by Theorem 3.6. \square

Now we study the geometric interpretation of the isotonized nets \mathcal{B}_{\pm} .

We have seen that any double cone \mathcal{O} in dS^2 can be represented as $\mathcal{O} = I_+ \times I_-$, where $I_{\pm} = \varphi_{\pm}(\mathcal{O})$. Then we may define the *horizon components* of a net \mathcal{A} on dS^2 as the nets on S^1 given by

$$(5.18) \quad \mathcal{A}_+(I) = \bigcap_{\mathcal{O}: \varphi_+(\mathcal{O}) \supset I} \mathcal{A}(\mathcal{O}) \quad , \quad \mathcal{A}_-(I) = \bigcap_{\mathcal{O}: \varphi_-(\mathcal{O}) \supset I} \mathcal{A}(\mathcal{O}).$$

Theorem 5.12. *Let \mathcal{A} be a Haag dual net on dS^2 , \mathcal{B} the corresponding pseudonet on S^1 . Then horizon components correspond to isotonized nets:*

$$(5.19) \quad \mathcal{A}_{\pm}(I) = \mathcal{B}_{\pm}(I).$$

As a consequence the horizon components are conformal nets.

Proof. Since \mathcal{A} is Haag dual, the chiral components may be equivalently defined as

$$(5.20) \quad \mathcal{A}_+(I) = \bigcap_{W: \varphi_+(W) \supset I} \mathcal{A}(W) \quad , \quad \mathcal{A}_-(I) = \bigcap_{W: \varphi_-(W) \supset I} \mathcal{A}(W),$$

and the equality (5.19) follows by eq. (5.13). \square

Remark 5.13. We could have also defined the horizon restriction net for any component \mathfrak{H}_\pm of the cosmological horizon, simply setting $\cap_{W \supset \psi_\pm(I)} \mathcal{A}(W)$, for any interval $I \subset \mathbb{R}$. In general it is a larger subnet than the horizon component \mathcal{A}_\pm .

Then we consider the conformal net on dS^2 given by

$$(5.21) \quad \mathcal{A}_\chi(I_+ \times I_-) = \mathcal{A}_+(I_+) \vee \mathcal{A}_-(I_-).$$

Theorem 5.14. \mathcal{A}_χ is a conformal expected subnet of \mathcal{A} , satisfying

$$(5.22) \quad \mathcal{A}_\chi(I_+ \times I_-) = \mathcal{A}_+(I_+) \otimes \mathcal{A}_-(I_-).$$

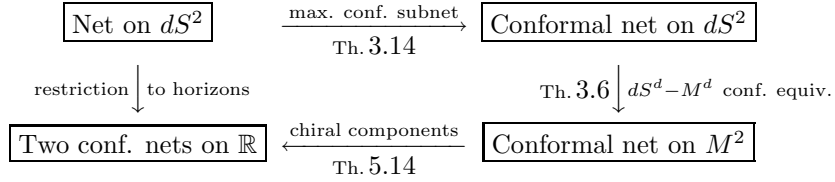
Indeed it is the chiral subnet of the maximal conformal expected subnet of \mathcal{A} .

Proof. The tensor product splitting follows by Proposition 5.9. As \mathcal{A}_χ is chiral conformal, it is immediate that it satisfies the local time-slice property, hence it is a Haag dual conformal net. Thus it is expected by Prop. 2.7. \square

We shall say that a net \mathcal{A} is chiral if it coincides with its chiral subnet.

The subnets \mathcal{A}_\pm may be considered as the chiral components of \mathcal{A} . Indeed, they correspond to the two chiral nets on the lightlike rays for a conformal net on the two-dimensional Minkowski space. Therefore we shall say that \mathcal{A} is a *chiral net* if it coincides with \mathcal{A}_χ .

The following table summarizes the chirality structure.



We conclude this section characterizing the chiral nets on dS^2 with only one horizon component.

Theorem 5.15. Let \mathcal{A} be a local net of von Neumann algebras on the de Sitter spacetime such that H is positive, resp. negative, where H is the generator of the rotation subgroup. Then the associated pseudonet \mathcal{B} , resp $\overline{\mathcal{B}}$ is indeed a local net, which holographically reconstructs \mathcal{A} : $\mathcal{A}(\mathcal{O}) = \mathcal{B}(\varphi_\pm(\mathcal{O}))$. In particular \mathcal{A} is conformal.

Proof. If H is positive, the pseudonet is isotonic, by 5.8 (iii). Analogously, if H is negative, the pseudonet is anti-isotonic, hence $\overline{\mathcal{B}}$ is isotonic. In both cases \mathcal{A} is chiral, hence conformal. \square

Corollary 5.16. The following are equivalent:

- The representation U has positive (resp. negative) energy
- $\mathcal{B}_-(I)$ (resp. $\mathcal{B}_+(I)$) is trivial and $\mathcal{A}_\chi(\mathcal{O}) = \mathcal{A}(\mathcal{O})$
- $\mathcal{A}(I_+ \times I_-) = \mathcal{A}_+(I_+)$ (resp. $= \mathcal{A}_-(I_-)$)
- \mathcal{A} is conformal and the translations on \mathfrak{H}_- (resp. on \mathfrak{H}_+) are trivial

Proof. Immediate by the above discussion. \square

We end up with a “holographic” dictionary:

| | |
|---|--|
| dS^2 | S^1 |
| wedge W | interval I |
| double cone \mathcal{O} | pair of intervals $I_1 \subset I_2$ |
| Haag dual nets \mathcal{A} on dS^2 | local pseudonets \mathcal{B} on S^1 |
| $SO(2, 1)$ covariance | Möbius covariance |
| horizon (chiral) components \mathcal{A}_\pm | isotonized nets \mathcal{B}_\pm |
| Reeh-Schlieder property for \mathcal{O} | intersection cyclicity |
| conformal invariance | property (5.16) |
| positive (negative) energy for \mathcal{A} | isotony (anti-isotony) for \mathcal{B} |
| chirality | $\mathcal{B} = \mathcal{B}_+ \otimes \overline{\mathcal{B}_-}$ |

6. FINAL COMMENTS

Equivalence principle and dethermalization. As is well known, Einstein equivalence principle is a fundamental guiding principle in General Relativity, although it is valid only at the infinitesimal (i.e. local) level. However, if one considers quantum effects, one may notice a certain asymmetry, yet between inertial observers in different spacetimes: the one in de Sitter spacetime feels the Gibbons-Hawking temperature, while the one in Minkowski spacetime is in a ground state. One way to describe the dethermalization effect is to say that it “restores” the symmetry: being a quantum effects, it needs a quantum (i.e. noncommutative) description. Only in the limit case where QFT becomes conformal (a situation closer to general covariance in classical general relativity) the dethermalization effect is described by classical flows. In the general case the noncommutative geometry is encoded in the net of local algebras (that takes the place of function algebras) and the dynamics is expressed in terms of this net.

Other spacetimes. Although this paper has dealt essentially with de Sitter spacetime, a good part of our description obviously holds in more general spacetimes. As mentioned, several spacetimes are conformal to subregions of Einstein static universe. For a d -dimensional spacetime \mathcal{M} in this class one can obviously extend the analysis made in the dS^d case: one can set up a correspondence between local conformal nets on \mathcal{M} and on M^d , hence providing a KMS characterization of the conformal vacuum on \mathcal{M} , and finding the evolutions corresponding to dethermalized observers. However, the partial geometric property of the noncommutative flow with positive energy is established only in dS^d case by using the large group of isometries of dS^d .

In particular we may consider a Robertson-Walker spacetime RW^d . In the positive curvature case, RW^d is $\mathbb{R} \times S^{d-1}$ with metric

$$ds^2 = dt^2 - f(t)^2 d\sigma^2,$$

where $d\sigma^2$ is the metric on the unit sphere S^{d-1} and $f(t) > 0$ (in the general case S^{d-1} is a manifold of constant curvature $K = 1, -1, 0$.) In this case we may also use the method of transplantation given in [14].

Classification. Recently [34], diffeomorphism covariant local nets on the two-dimensional Minkowski spacetime, with central charge less than one, have been completely classified. By the conformal equivalence Th. 3.6 one immediately translates this result on dS^2 , namely one has a classification of the two-dimensional diffeomorphism covariant local nets on dS^2 with central charge less than one.

Models, modular localization. The methods in [12] provide a construction of (free) local nets on dS^2 associated with unitary representations of the de Sitter group $SO_0(d, 1)$, and conformal nets on S^1 associated with unitary representations

of $PSL(2, \mathbb{R})$. The isomorphism between $SO_0(2, 1)$ and $PSL(2, \mathbb{R})$ gives the holography in these models and is at the basis of our general analysis.

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